QUANTIZATION OF POISSON GROUPS

F. Gavarini

Dipartimento di Matematica, Istituto G. Castelnuovo,
Università degli studi di Roma "La Sapienza"

Abstract. The quantization of well-known pairs of Poisson groups of a wide class is studied by means of Drinfeld's double construction and dualization via formal Hopf algebras; new quantized enveloping algebras $U_{q,\varphi}^{M,\infty}(\mathfrak{h})$ and quantum formal groups $F_{q,\varphi}^{M,\infty}[G]$ are introduced (both in one-parameter and multi-parameter case), and their specializations at roots of 1 (in particular, their classical limits) are studied: a new insight into the classical Poisson duality is thus obtained.

Introduction

"Dualitas dualitatum
et omnia dualitas"
N. Barbecue, "Scholia"

Let G be a semisimple, connected, and simply connected affine algebraic group over an algebraically closed field of characteristic zero; we consider a family of structures of Poisson group on G, indexed by a multiparameter τ, which generalize the well-known Sklyanin-Drinfeld one (cf. e. g. [DP], §11). Then every such Poisson group G^τ has a well defined corresponding dual Poisson group H^τ, and $\mathfrak{g}^\tau := Lie(G^\tau)$ and $\mathfrak{h}^\tau := Lie(H^\tau)$ are Lie bialgebras dual of each other.

In 1985 Drinfeld ([Dr]) and Jimbo ([Ji]) provided a quantization $U_q^Q(\mathfrak{g})$ of $U(\mathfrak{g}) = U(\mathfrak{g}^0)$, namely a Hopf algebra $U_q^Q(\mathfrak{g})$ over $k(q)$, presented by generators and relations with a $k[q,q^{-1}]$-form $\widehat{U}_q^Q(\mathfrak{g})$ which for $q \to 1$ specializes to $U(\mathfrak{g})$ as a Poisson Hopf coalgebra. This has been extended to general parameter τ introducing multiparameter quantum groups $U_{q,\varphi}^Q(\mathfrak{g})$ (cf. [R], and [CV-1], [CV-2]). Dually, by means of a Peter-Weyl type axiomatic trick one constructs a Hopf algebra $F_q^P[G]$ of matrix coefficients of $U_q^Q(\mathfrak{g})$ with a $k[q,q^{-1}]$-form $\widehat{F}_q^P[G]$ which specializes to $F[G]$, as a Poisson Hopf algebra, for $q \to 1$; in particular $\widehat{F}_q^P[G]$ is nothing but the Hopf subalgebra of "functions" in $F_q^P[G]$ which take values in

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$k[q,q^{-1}]$ when "evaluated" on $\widehat{U}_q^Q(\mathfrak{g})$ (in a word, the $k[q,q^{-1}]$-integer valued functions on $\widehat{U}_q^Q(\mathfrak{g})$). This again extends to general τ (cf. [CV-2]).

So far the quantization procedure only dealt with the Poisson group G^τ; the dual group H is involved defining a different $k[q,q^{-1}]$-form $\widetilde{U}_q^P(\mathfrak{g})$ (of a quantum group $U_q^P(\mathfrak{g})$) which specializes to $F[H]$ (as a Poisson Hopf algebra) for $q \rightarrow 1$ (cf. [DP]), with generalization to the general case possible again. Here a new phenomenon of "crossing dualities" (duality among enveloping and function algebra and duality among Poisson groups) occurs which was described (in a formal setting) by Drinfel'd (cf. [Dr], §7). This leads to consider the following: let $F_q^Q[G]$ be the quantum function algebra dual of $U_q^P(\mathfrak{g})$, and look at the "dual" of $\widetilde{U}_q^P(\mathfrak{g})$ within $F_q^Q[G]$, call it $\widetilde{F}_q^Q[G]$, which should be the Hopf algebra of $k[q,q^{-1}]$-integer valued functions on $\widetilde{U}_q^P(\mathfrak{g})$; then this should specialize to $U(\mathfrak{h})$ (as a Poisson Hopf coalgebra) for $q \rightarrow 1$; the same conjecture can be formulated in the general case too.

The starting aim of the present work was to achieve this goal, i. e. to construct $F_q^Q[G]$ and its $k[q,q^{-1}]$-form $\widetilde{F}_q^Q[G]$, and to prove that $\widetilde{F}_q^Q[G]$ is a deformation of the Poisson Hopf coalgebra $U(\mathfrak{h})$. This goal is succesfully attained by performing a suitable dualization of Drinfeld's quantum double; but by the way, this leads to discover a new "quantum group" (both one-parameter and multiparameter), which we call $U_q^{M,\infty}(\mathfrak{h})$, which is to $U(\mathfrak{h})$ as $U_q^M(\mathfrak{g})$ is to $U(\mathfrak{g})$; in particular it has an integer form $\widehat{U}_q^Q(\mathfrak{h})$ which is a quantization of $U(\mathfrak{h})$, and an integer form $\widetilde{U}_q^{P,\infty}(\mathfrak{h})$ which is a quantization of $F^\infty[G]$ (the function algebra of the formal Poisson group associated to G); this is also produced in the shape of a *quantum formal group* $\widehat{F}_q^{P,\infty}[G]$. Moreover, we exhibit a quantum analog of the Hopf pairings $F[H] \otimes U(\mathfrak{h}) \rightarrow k$, $F[G] \otimes U(\mathfrak{g}) \rightarrow k$, $F^\infty[G] \otimes U(\mathfrak{g}) \rightarrow k$, and also a quantum analog of the Lie bialgebra pairing $\mathfrak{h} \otimes \mathfrak{g} \rightarrow k$, which are strictly related with each other and with the already known quantum pairing $F_q^P[G] \otimes U_q^Q(\mathfrak{g}) \rightarrow k(q)$: all this is stressed by introducing the definition of *quantum Poisson pairing*. Once again all this extend to the multiparameter case. Thus in particular we provide a quantization for a wide class of Poisson groups (the H^τ's); now, in the summer of 1995 (when the present work was already accomplished) a quantization of any Poisson group was presented in [EK-1] and [EK-2], but greatest generality implies lack of concreteness: in contrast, our construction is extremely concrete, by hands; moreover, it allows specialization at roots of 1, construction of quantum Frobenius morphisms, and so on (like for $U_q^Q(\mathfrak{g})$ and $U_q^P(\mathfrak{g})$), which is not possible in the approach of [EK-1], [EK-2].

Finally, a brief sketch of the main ideas of the paper. Our aim being to study the "dual" of a quantum group $U_{q,\varphi}^M(\mathfrak{g})$ (M being a lattice of weights), we proceed as follows. First, we select as operation of "dualization" (of a Hopf algebra H) the most naïve one, namely taking the *full linear dual* (i. e. H^*, rather than the usual Hopf dual H°), the latter being a *formal* Hopf algebra (rather than a common Hopf algebra). Second, we remark that $U_{q,\varphi}^M(\mathfrak{g})$ is a quotient, as a Hopf algebra, of a quantum double $D_{q,\varphi}^M(\mathfrak{g}) := D\left(U_{q,\varphi}^Q(\mathfrak{b}_-), U_{q,\varphi}^M(\mathfrak{b}_+), \pi_+^\varphi\right)$ (cf. §3), hence its linear dual $U_{q,\varphi}^M(\mathfrak{g})^*$ embeds into the formal Hopf algebra $D_{q,\varphi}^M(\mathfrak{g})^*$. Third, since $D_{q,\varphi}^M(\mathfrak{g}) \cong U_{q,\varphi}^M(\mathfrak{b}_+) \otimes U_{q,\varphi}^Q(\mathfrak{b}_-)$ (as coalgebras) we have $D_{q,\varphi}^M(\mathfrak{g})^* \cong U_{q,\varphi}^M(\mathfrak{b}_+)^* \,\widehat{\otimes}\, U_{q,\varphi}^Q(\mathfrak{b}_-)^*$ (as algebras), where $\widehat{\otimes}$ denotes topological tensor product. Fourth, DRT pairings among quantum Borel algebras yield natural embeddings $U_{q,\varphi}^M(\mathfrak{b}_\pm) \hookrightarrow U_{q,\varphi}^{M'}(\mathfrak{b}_\mp)^*$ (M' being the dual lattice of M). Fifth, from previous remarks we locate a special formal Hopf subalgebra of $\left(U_{q,\varphi}^{M'}(\mathfrak{b}_-) \,\widehat{\otimes}\, U_{q,\varphi}^P(\mathfrak{b}_+)\right) \subseteq D_{q,\varphi}^M(\mathfrak{g})^*$.

Seventh, we take out of this an axiomatic construction of a formal Hopf algebra $U_{q,\varphi}^{M'}(\mathfrak{h})$, by definition isomorphic to the one of step six and therefore naturally paired with $U_{q,\varphi}^{M}(\mathfrak{g})$: this is the object we were looking for, sort of completion of $F_{q,\varphi}^{M'}[G]$, and from its very construction the whole series of results we stated above will be proved with no serious trouble.

Sections 1 to 4 introduce the already known material; the original part of the paper is sections 5 to 8. The Appendix describes in full detail the example $G = SL(2)$.

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§ 1 The classical objects

1.1 Cartan data. Let $A := (a_{ij})_{i,j=1,\dots,n}$ be a $n \times n$ symmetrizable Cartan matrix; thus we have $a_{ij} \in \mathbb{Z}$ with $a_{ii} = 2$ and $a_{ij} \leq 0$ if $i \neq j$, and there exists a vector $(d_1, \dots, d_n)$ with relatively prime positive integral entries d_i such that $(d_i a_{ij})_{i,j=1,\dots,n}$ is a symmetric positive definite matrix. Define the *weight lattice* P to be the lattice with basis $\{\omega_1, \dots, \omega_n\}$ (the *fundamental weights*); let $P_+ = \sum_{i=1}^n \mathbb{N}\omega_i$ be the subset of *dominant integral weights*, let $\alpha_j = \sum_{i=1}^n a_{ij}\omega_i$ $(j = 1, \dots, n)$ be the *simple roots*, and $Q = \sum_{j=1}^n \mathbb{Z}\alpha_j$ $(\subset P)$ the *root lattice*, $Q_+ = \sum_{j=1}^n \mathbb{N}\alpha_j$ the *positive root lattice*. Let W be the Weyl group associated to A, with generators $s_1, \dots, s_n$, and let $\Pi := \{\alpha_1, \dots, \alpha_n\}$: then $R := W\Pi$ is the set of *roots*, $R^+ = R \cap Q_+$ the set of *positive roots*; finally, we set $N := \#(R^+)$ $(= |W|)$.

Define bilinear pairings $\langle\ |\ \rangle \colon Q \times P \to \mathbb{Z}$ and $(\ |\ ) \colon Q \times P \to \mathbb{Z}$ by $\langle \alpha_i | \omega_j \rangle = \delta_{ij}$ and $(\alpha_i | \omega_j) = \delta_{ij} d_i$. Then $(\alpha_i | \alpha_j) = d_i a_{ij}$, giving a symmetric $\mathbb{Z}$-valued W-invariant bilinear form on Q such that $(\alpha|\alpha) \in 2\mathbb{Z}$. We can also extend the $\mathbb{Z}$-bilinear pairing $(\ |\ ) \colon Q \times P \to \mathbb{Z}$ to a (non-degenerate) pairing $(\ |\ ) \colon (\mathbb{Q} \otimes_{\mathbb{Z}} Q) \times (\mathbb{Q} \otimes_{\mathbb{Z}} P) \to \mathbb{Z}$ of $\mathbb{Q}$-vector spaces by scalar extension: restriction gives a pairing $(\ |\ ) \colon P \times P \to \mathbb{Q}$ (looking at P as a sublattice of $\mathbb{Q} \otimes_{\mathbb{Z}} Q$), which takes values in $\mathbb{Z}[d^{-1}]$, where $d := det\left((a_{ij})_{i,j=1}^n\right)$. Given a pair of lattices (M, M'), with $Q \leq M, M' \leq P$, we say that they are *dual of each other* if

$$M' = \{\, y \in P \mid (M, y) \subseteq \mathbb{Z} \,\} \quad , \qquad M = \{\, x \in P \mid (x, M') \subseteq \mathbb{Z} \,\}$$

the two conditions being equivalent; then for any lattice M with $Q \leq M \leq P$ there exists a unique dual lattice M' such that $Q \leq M' \leq P$.

1.2 The Poisson groups G and H. Let G be a connected simply-connected semisimple affine algebraic group over an algebraically closed field k of charachteristic 0. Fix a maximal torus $T \leq G$ and opposite Borel subgroups $B_\pm$, with unipotent subgroups $U_\pm$, such that $B_+ \cap B_- = T$, and let $\mathfrak{g} := Lie(G)$, $\mathfrak{t} := Lie(T)$, $\mathfrak{b}_\pm := Lie(B_\pm)$, $\mathfrak{n}_\pm := Lie(U_\pm)$; fix also $\tau := (\tau_1, \dots, \tau_n) \in Q^n$ such that $(\tau_i, \alpha_j) = -(\tau_j, \alpha_i)$ for all $i, j = 1, \dots, n$: when $\tau = (0, \dots, 0)$ we shall simply skip it throughout. Set $K = G \times G$, define $G^\tau := G$ embedded in K as the diagonal subgroup, and define a second subgroup

$$H^\tau := \{\, (u_- t_-, t_+ u_+) \mid u_\pm \in U_\pm, t_\pm \in T, t_- t_+ \in \exp(\mathfrak{t}^\tau) \,\} \; (\leq B_- \times B_+ \leq K)$$

where $\mathfrak{t}^\tau := \sum_{i=1}^n k \cdot h_{-\alpha_i+2\tau_i} \oplus h_{\alpha_i+2\tau_i} \leq \mathfrak{t} \oplus \mathfrak{t} \leq \mathfrak{g} \oplus \mathfrak{g} = \mathfrak{k} := Lie(K)$; thus

$$\mathfrak{h}^\tau := Lie(H^\tau) = (\mathfrak{n}_-, 0) \oplus \mathfrak{t}^\tau \oplus (0, \mathfrak{n}_+) .$$

Thus we have a triple (K, G^τ, H^τ); this is an algebraic Manin triple (i. e. its "tangent triple" $(\mathfrak{k}, \mathfrak{g}^\tau, \mathfrak{h}^\tau)$ is a Manin triple), when a non-degenerate symmetric invariant bilinear form on $\mathfrak{k}$ is defined as follows: first rescale the Killing form $(\ ,\ ) \colon \mathfrak{g} \otimes \mathfrak{g} \to k$ so that short roots of $\mathfrak{g}$ have square length 2; then define the form on $\mathfrak{k} = \mathfrak{g} \oplus \mathfrak{g}$ by

$$\langle x_1 \oplus y_1, x_2 \oplus y_2 \rangle := \frac{1}{2}(y_1, y_2) - \frac{1}{2}(x_1, x_2) .$$

In general, if $(\mathfrak{k}', \mathfrak{g}', \mathfrak{h}')$ is any Manin triple, the bilinear form on $\mathfrak{k}'$ gives by restriction a non-degenerate pairing $\langle\ ,\ \rangle \colon \mathfrak{h}' \otimes \mathfrak{g}' \to k$ which is a pairing of Lie bialgebras, that is

$$\langle x, [y_1, y_2] \rangle = \langle \delta_{\mathfrak{h}}(x), y_1 \wedge y_2 \rangle , \quad \langle [x_1, x_2], y \rangle = \langle x_1 \wedge x_2, \delta_{\mathfrak{g}}(y) \rangle$$

for all $x, x_1, x_2 \in \mathfrak{h}'$, $y, y_1, y_2 \in \mathfrak{g}'$, where $\delta_{\mathfrak{h}'}$, resp. $\delta_{\mathfrak{g}'}$, is the Lie cobracket of $\mathfrak{h}'$, resp. $\mathfrak{g}'$; we shall call such pairing also *Poisson pairing*, and denote it by $\pi_{\mathcal{P}}(h, g) := \langle h, g \rangle$ for all $h \in \mathfrak{h}'$, $g \in \mathfrak{g}'$. In the present case the Poisson pairing is described by formulas

$$\begin{array}{ccc} \langle \mathrm{f}_i^\tau, f_j \rangle = 0 & \langle \mathrm{f}_i^\tau, h_j \rangle = 0 & \langle \mathrm{f}_i^\tau, e_j \rangle = -\frac{1}{2}\delta_{ij} d_i^{-1} \\ \langle \mathrm{h}_i^\tau, f_j \rangle = 0 & \langle \mathrm{h}_i^\tau, h_j \rangle = a_{ij} d_j^{-1} = a_{ji} d_i^{-1} & \langle \mathrm{h}_i^\tau, e_j \rangle = 0 \\ \langle \mathrm{e}_i^\tau, f_j \rangle = \frac{1}{2}\delta_{ij} d_i^{-1} & \langle \mathrm{e}_i^\tau, h_j \rangle = 0 & \langle \mathrm{e}_i^\tau, e_j \rangle = 0 \end{array} \qquad (1.1-a)$$

where the f_s^τ, h_s^τ, e_s^τ $(s = 1, \dots, n)$, resp. f_s, h_s, e_s $(s = 1, \dots, n)$, are Chevalley-type generators of $\mathfrak{h}^\tau$, resp. $\mathfrak{g}^\tau$, embedded inside $\mathfrak{k} = \mathfrak{g} \oplus \mathfrak{g}$, namely $\mathrm{f}_s^\tau = f_s \oplus 0$, $\mathrm{h}_s^\tau = h_{-\alpha_s+2\tau_s} \oplus h_{\alpha_s+2\tau_s}$, $\mathrm{e}_s^\tau = 0 \oplus e_s$, and $f_s = f_s \oplus f_s$, $h_s = h_s \oplus h_s$, $e_s = e_s \oplus e_s$ (see below); moreover we also record that

$$\langle \mathrm{e}_\alpha^\tau, f_\beta \rangle = \frac{1}{2}\delta_{\alpha\beta} d_\alpha^{-1} , \qquad \langle \mathrm{f}_\alpha^\tau, e_\beta \rangle = -\frac{1}{2}\delta_{\alpha\beta} d_\alpha^{-1} \qquad (1.1-b)$$

for all $\alpha, \beta \in R^+$, where $\mathrm{e}_\alpha^\tau, \mathrm{f}_\alpha, e_\beta, f_\beta$ are Chevalley-type generators of $(\mathfrak{h}^\tau)_\alpha$, $(\mathfrak{h}^\tau)_{-\alpha}$, $(\mathfrak{g}^\tau)_\beta$, $(\mathfrak{g}^\tau)_{-\beta}$ and $d_\alpha := \frac{(\alpha,\alpha)}{2}$ for all $\alpha \in R^+$ (in particular $d_{\alpha_i} = d_i \ \ \forall\, i = 1, \dots, n$).[1]

1.3 The Poisson Hopf coalgebra $U(\mathfrak{g}^\tau)$. It is known that the universal enveloping algebra $U(\mathfrak{g}^\tau) = U(\mathfrak{g})$ can be presented as the associative k-algebra with 1 generated by elements , f_i, h_i, e_i $(i = 1, \dots, n)$ (the *Chevalley generators*) satisfying the well-known Serre's relations (cf. for instance [Hu], ch. V). As an enveloping algebra of a Lie algebra, $U(\mathfrak{g}^\tau)$ has a canonical structure of Hopf algebra, given by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $S(x) = -x$, $\varepsilon(x) = 0$ for all $x \in \mathfrak{g}$; finally, the Poisson structure of G^τ reflects into a Lie coalgebra structure $\delta = \delta_{\mathfrak{g}^\tau} \colon \mathfrak{g}^\tau \longrightarrow \mathfrak{g}^\tau \otimes \mathfrak{g}^\tau$ of $\mathfrak{g}^\tau$, extending to a co-Poisson structure $\delta \colon U(\mathfrak{g}^\tau) \longrightarrow U(\mathfrak{g}^\tau) \otimes U(\mathfrak{g}^\tau)$ (compatible with the Hopf structure) given by (cf. [DL], §8, and [DKP], §7.7)

$$\begin{aligned} \delta(f_i) &= \frac{(\alpha_i + 2\tau_i | \alpha_i + 2\tau_i)}{2}\, h_{\alpha_i+2\tau} \otimes f_i - \frac{(\alpha_i + 2\tau_i | \alpha_i + 2\tau_i)}{2}\, f_i \otimes h_{\alpha_i+2\tau} \\ \delta(h_i) &= 0 \\ \delta(e_i) &= \frac{(\alpha_i - 2\tau_i | \alpha_i - 2\tau_i)}{2}\, h_{\alpha_i-2\tau} \otimes e_i - \frac{(\alpha_i - 2\tau_i | \alpha_i - 2\tau_i)}{2}\, e_i \otimes h_{\alpha_i-2\tau} \ . \end{aligned} \qquad (1.2)$$

[1] *Warning:* beware, in particular, of the normalization we chose for the symmetric form of $\mathfrak{k}$, which is different (by a coefficient $\frac{1}{2}$) from the one fixed in [DP].

1.4 The Poisson Hopf coalgebra $U(\mathfrak{h}^\tau)$. We already remarked that $\mathfrak{h}^\tau \cong \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+ \cong \mathfrak{n}_- \oplus \mathfrak{t}^\tau \oplus \mathfrak{n}_+$ as vector spaces: then the Lie structure of $\mathfrak{h}^\tau$ is uniquely determined by the following constraints: *(a)* $\mathfrak{n}_-$, $\mathfrak{t}^\tau$, $\mathfrak{n}_+$ are Lie subalgebras of $\mathfrak{h}^\tau$; *(b)* $[n_+, n_-] = 0 \;\forall\, n_+ \in \mathfrak{n}_+, n_- \in \mathfrak{n}_-$; *(c)* $[\mathrm{h}_i^\tau, n_-] = -\langle \alpha_i - 2\tau_i, \beta \rangle\, n_- \;\; \forall\, i = 1, \dots, n, n_- \in (\mathfrak{n}_-)_\beta, \beta \in R^-$; *(d)* $[\mathrm{h}_i^\tau, n_+] = \langle \alpha_i + 2\tau_i, \beta \rangle\, n_+ \;\forall\, i = 1, \dots, n, n_+ \in (\mathfrak{n}_+)_\alpha, \alpha \in R^+$.

Thus if f_i^τ, h_i^τ, e_i^τ $(i = 1, \dots, n)$ are Chevalley-type generators of $\mathfrak{g}$ (thought of as elements of $\mathfrak{n}_-$, $\mathfrak{t} \cong \mathfrak{t}^\tau$, $\mathfrak{n}_+$), $U(\mathfrak{h}^\tau)$ can be presented as the associative k-algebra with 1 generated by f_i^τ, h_i^τ, e_i^τ $(i = 1, \dots, n)$ with relations

$$\begin{gathered}
\mathrm{h}_i^\tau \mathrm{h}_j^\tau - \mathrm{h}_j^\tau \mathrm{h}_i^\tau = 0\,, \qquad \mathrm{e}_i^\tau \mathrm{f}_j^\tau - \mathrm{f}_j^\tau \mathrm{e}_i^\tau = 0 \\
\mathrm{h}_i^\tau \mathrm{f}_j^\tau - \mathrm{f}_j^\tau \mathrm{h}_i^\tau = \langle \alpha_i - 2\tau_i, \alpha_j \rangle \mathrm{f}_j^\tau\,, \qquad \mathrm{h}_i^\tau \mathrm{e}_j^\tau - \mathrm{e}_j^\tau \mathrm{h}_i^\tau = \langle \alpha_i + 2\tau_i, \alpha_j \rangle \mathrm{e}_j^\tau \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} (\mathrm{f}_i^\tau)^{1-a_{ij}-k} \mathrm{f}_j^\tau (\mathrm{f}_i^\tau)^k = 0 \qquad (i \neq j) \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} (\mathrm{e}_i^\tau)^{1-a_{ij}-k} \mathrm{e}_j^\tau (\mathrm{e}_i^\tau)^k = 0 \qquad (i \neq j)
\end{gathered} \tag{1.3}$$

for all $i, j = 1, \dots, n$; its natural Hopf structure is given by

$$\begin{aligned}
\Delta(\mathrm{f}_i^\tau) &= \mathrm{f}_i^\tau \otimes 1 + 1 \otimes \mathrm{f}_i^\tau\,, \quad S(\mathrm{f}_i^\tau) = -\mathrm{f}_i^\tau\,, \quad \varepsilon(\mathrm{f}_i^\tau) = 0 \\
\Delta(\mathrm{h}_i^\tau) &= \mathrm{h}_i^\tau \otimes 1 + 1 \otimes \mathrm{h}_i^\tau\,, \quad S(\mathrm{h}_i^\tau) = -\mathrm{h}_i^\tau\,, \quad \varepsilon(\mathrm{h}_i^\tau) = 0 \\
\Delta(\mathrm{e}_i^\tau) &= \mathrm{e}_i^\tau \otimes 1 + 1 \otimes \mathrm{e}_i^\tau\,, \quad S(\mathrm{e}_i^\tau) = -\mathrm{e}_i^\tau\,, \quad \varepsilon(\mathrm{e}_i^\tau) = 0\,.
\end{aligned} \tag{1.4}$$

and the co-Poisson structure $\delta = \delta_{\mathfrak{h}^\tau} \colon U(\mathfrak{h}^\tau) \longrightarrow U(\mathfrak{h}^\tau) \otimes U(\mathfrak{h}^\tau)$ by

$$\begin{aligned}
\delta(\mathrm{f}_i^\tau) &= d_i \mathrm{h}_i^\tau \otimes \mathrm{f}_i^\tau - d_i \mathrm{f}_i^\tau \otimes \mathrm{h}_i^\tau \\
\delta(\mathrm{h}_i^\tau) &= 4\, d_i^{-1} \cdot \sum_{\gamma \in R^+} d_\gamma (\gamma | \alpha_i) \cdot (\mathrm{e}_\gamma^\tau \otimes \mathrm{f}_\gamma^\tau - \mathrm{f}_\gamma^\tau \otimes \mathrm{e}_\gamma^\tau) \\
\delta(\mathrm{e}_i^\tau) &= d_i \mathrm{e}_i^\tau \otimes \mathrm{h}_i^\tau - d_i \mathrm{h}_i^\tau \otimes \mathrm{e}_i^\tau\,.
\end{aligned} \tag{1.5}$$

§ 2 Quantum Borel algebras and DRT pairings

2.1 Notations. We shall introduce our quantum groups using the construction of [DL] and [CV-1], [CV-2]. For all $s, n \in \mathbb{N}$, let $(n)_q := \frac{q^n - 1}{q - 1}$ $(\in k[q])$, $(n)_q! := \prod_{r=1}^n (r)_q$, $\binom{n}{s}_q := \frac{(n)_q!}{(s)_q!(n-s)_q!}$ $(\in k[q])$, and $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$ $(\in k[q, q^{-1}])$, $[n]_q! := \prod_{r=1}^n [r]_q$, $\left[{n \atop s}\right]_q := \frac{[n]_q!}{[s]_q![n-s]_q!}$ $(\in k[q, q^{-1}])$; let $q_\alpha := q^{d_\alpha}$ for all $\alpha \in R^+$, and $q_i := q_{\alpha_i}$. Let Q, P be as in §1; we fix an endomorphism φ of the $\mathbb{Q}$-vector space $\mathbb{Q}P := \mathbb{Q} \otimes_{\mathbb{Z}} P$ which is antisymmetric — with respect to $(\;|\;)$ — and satisfies the conditions

$$\varphi(Q) \subseteq Q\,, \quad \tfrac{1}{2}(\varphi(P) \mid P) \subseteq \mathbb{Z}\,, \quad 2AYA^{-1} \in M_n(\mathbb{Z})$$

where, letting $\tau_i := \frac{1}{2}\varphi(\alpha_i) = \sum_{j=1}^{n} y_{ji}\alpha_j$, we set $Y := (y_{ij})_{i,j=1,\dots,n;}$, $M_n(\mathbb{Z})$ is the set of $n \times n$-matrices with integer entries, and A is our Cartan matrix. For later use, we also define

$$\tau_\alpha := \frac{1}{2}\varphi(\alpha)\,, \quad \tau_i := \tau_{\alpha_i} \tag{2.1}$$

for all $\alpha \in R$, $i = 1,\dots,n$. It is proved in [CV-1] that $(\mathrm{id}_{\mathbb{Q}P} + \varphi)$ and $(\mathrm{id}_{\mathbb{Q}P} - \varphi)$ are isomorphisms, adjoint of each other with respect to $(\ \mid\ )$: then we define $r := (\mathrm{id}_{\mathbb{Q}P} + \varphi)^{-1}$, $\overline{r} := (\mathrm{id}_{\mathbb{Q}P} - \varphi)^{-1}$.

2.2 Quantum Borel algebras. Let M be any lattice such that $Q \leq M \leq P$; then (cf. [DL] and [CV-1]) $U_{q,\varphi}^M(\mathfrak{b}_-)$ (resp. $U_{q,\varphi}^M(\mathfrak{b}_+)$) is the associative $k(q)$-algebra with 1 generated by $\{ L_\mu \mid \mu \in M \} \cup \{ F_i \mid i = 1,\dots,n \}$ (resp. $\{ L_\mu \mid \mu \in M \} \cup \{ E_i \mid i = 1,\dots,n \}$) with relations

$$L_0 = 1\,, \quad L_\mu L_\nu = L_{\mu+\nu}\,,$$
$$L_\mu F_j = q^{-(\mu|\alpha_j)} F_j L_\mu\,, \qquad \sum_{p+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} F_i^p F_j F_i^s = 0$$
$$(\text{resp.} \quad L_\mu E_j = q^{(\mu|\alpha_j)} E_j L_\mu\,, \qquad \sum_{p+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} E_i^p E_j E_i^s = 0\,)$$

for all $i,j = 1,\dots,n$ and $\mu,\nu \in M$; furthermore, it is proved in [CV-1] that $U_{q,\varphi}^M(\mathfrak{b}_-)$ and $U_{q,\varphi}^M(\mathfrak{b}_-)$ can be endowed with a Hopf algebra structure given by[2]

$$\Delta_\varphi(F_i) = F_i \otimes L_{-\alpha_i-\tau_i} + L_{\tau_i} \otimes F_i\,, \quad \varepsilon_\varphi(F_i) = 0\,, \quad S_\varphi(F_i) = -F_i L_{\alpha_i}$$
$$\Delta_\varphi(L_\mu) = L_\mu \otimes L_\mu\,, \quad \varepsilon_\varphi(L_\mu) = 1\,, \quad S_\varphi(L_\mu) = L_{-\mu}$$
$$\Delta_\varphi(E_i) = E_i \otimes L_{\tau_i} + L_{\alpha_i-\tau_i} \otimes E_i\,, \quad \varepsilon_\varphi(E_i) = 0\,, \quad S_\varphi(E_i) = -L_{-\alpha_i} E_i$$

for all $i = 1,\dots,n$, $\mu \in M$. In particular we use notation $K_\alpha := L_\alpha \ \forall\, \alpha \in Q$. We shall also consider the subalgebras (of quantum Borel algebras) $U_{q,\varphi}^M(\mathfrak{t})$ (generated by the L_μ's), $U_{q,\varphi}(\mathfrak{n}_+)$ (generated by the E_i's), and $U_{q,\varphi}(\mathfrak{n}_-)$ (generated by the F_i's).

When $\varphi = 0$ we could skip the superscript φ, our quantum algebras then conciding with the one-parameter ones, i. e. those of, say, [Lu-1], [DP], etc.

2.3 DRT pairings. From now on, if H is any Hopf algebra, then H^{op} will denote the same coalgebra as H with the opposite multiplication, while H_{op} will denote the same algebra as H with the opposite comultiplication.

From [DL], §2 (for the 1-parameter case), and [CV-1], §3 (for the multiparameter case) we recall the existence of perfect (i. e. non-degenerate) pairings of Hopf algebras among quantum Borel algebras

$$\pi_-^\varphi\colon U_{q,\varphi}^P(\mathfrak{b}_-)_{op} \otimes U_{q,\varphi}^Q(\mathfrak{b}_+) \to k(q)\,, \quad \pi_+^\varphi\colon U_{q,\varphi}^Q(\mathfrak{b}_-)_{op} \otimes U_{q,\varphi}^P(\mathfrak{b}_+) \to k(q)$$
$$\overline{\pi_-^\varphi}\colon U_{q,\varphi}^P(\mathfrak{b}_+)_{op} \otimes U_{q,\varphi}^Q(\mathfrak{b}_-) k(q)\,, \quad \overline{\pi_+^\varphi}\colon U_{q,\varphi}^Q(\mathfrak{b}_+)_{op} \otimes U_{q,\varphi}^P(\mathfrak{b}_-) k(q)$$

[2] Actually, we use here a different normalization than in [CV-1]: the results therein coincide with the ones we list below up to change $q \leftrightarrow q^{-1}$, $L_\lambda \leftrightarrow L_{-\lambda}$ (or $K_\lambda \leftrightarrow K_{-\lambda}$).

which we select as given by

$$\pi_-^\varphi(L_\lambda,K_\alpha)=q^{-(r(\lambda)|\alpha)}\,,\ \pi_-^\varphi(L_\lambda,E_j)=0\,,\ \pi_-^\varphi(F_i,K_\alpha)=0\,,\ \pi_-^\varphi(F_i,E_j)=\delta_{ij}\frac{q^{-(r(\tau_i)|\tau_i)}}{(q_i^{-1}-q_i)}$$
$$\pi_+^\varphi(K_\alpha,L_\lambda)=q^{-(r(\alpha)|\lambda)}\,,\ \pi_+^\varphi(K_\alpha,E_j)=0\,,\ \pi_+^\varphi(F_i,L_\lambda)=0\,,\ \pi_+^\varphi(F_i,E_j)=\delta_{ij}\frac{q^{-(r(\tau_i)|\tau_i)}}{(q_i^{-1}-q_i)}$$
$$\overline{\pi_-^\varphi}(L_\lambda,K_\alpha)=q^{(r(\lambda)|\alpha)}\,,\ \overline{\pi_-^\varphi}(E_i,K_\alpha)=0\,,\ \overline{\pi_-^\varphi}(L_\lambda,F_j)=0\,,\ \overline{\pi_-^\varphi}(E_i,F_j)=\delta_{ij}\frac{q^{+(r(\tau_i)|\tau_i)}}{(q_i-q_i^{-1})}$$
$$\overline{\pi_+^\varphi}(K_\alpha,L_\lambda)=q^{(r(\alpha)|\lambda)}\,,\ \overline{\pi_+^\varphi}(E_i,K_\alpha)=0\,,\ \overline{\pi_+^\varphi}(L_\lambda,F_j)=0\,,\ \overline{\pi_+^\varphi}(E_i,F_j)=\delta_{ij}\frac{q^{+(r(\tau_i)|\tau_i)}}{(q_i-q_i^{-1})}$$
$$\forall\ i,j=1,\dots,n\,,\ \alpha\in Q\,,\ \lambda\in P\,.$$

Remark : Hopf pairings like those above were introduced by Drinfeld, Rosso, Tanisaki, and others, whence we shall call them *DRT pairings*; here again some differences occur with respect to [CV-1], [CV-2] (or even [DL] for the simplest case), because of different definition of the Hopf structure: cf. also [Tn] and [DD].

In the sequel if π is any DRT pairing we will also set $\langle x,y\rangle_\pi$ for $\pi(x,y)$.

2.4 PBW bases. It is known that both $U_{q,\varphi}^Q(\mathfrak{b}_-)$ and $U_{q,\varphi}^P(\mathfrak{b}_+)$ have bases of Poincaré-Birkhoff-Witt type (in short "PBW bases"): we fix (once and for all) any reduced expression of w_0 (the longest element in the Weyl group W), namely $w_0=s_{i_1}s_{i_2}\cdots s_{i_N}$ (with $N:=\#(R^+)=$ number of positive roots); thus we have a total convex ordering (cf. [Pa] and [DP], §8.2) $\alpha^1,\alpha^2:=s_{i_1}(\alpha_{i_2}),\dots,\alpha^N:=s_{i_1}s_{i_2}\cdots s_{i_{N-1}}(\alpha_{i_N})$, hence — following Lusztig and others: cf. e. g. [Lu-2] — we can construct root vectors E_{α^r}, $r=1,\dots,N$ as in [DP] or [CV-1] and get PBW bases of *increasing* ordered monomials $\left\{L_\mu\cdot\prod_{r=1}^N F_{\alpha^r}^{f_r}\,\middle|\,\mu\in M;f_1,\dots,f_N\in\mathbb{N}\right\}$ for $U_{q,\varphi}^M(\mathfrak{b}_-)$ and $\left\{L_\mu\cdot\prod_{r=1}^N E_{\alpha^r}^{e_r}\,\middle|\,\mu\in M;e_1,\dots,e_N\in\mathbb{N}\right\}$ for $U_{q,\varphi}^M(\mathfrak{b}_+)$ or similar PBW bases of *decreasing* ordered monomials.

Now, for every monomial $\mathcal{E}$ in the E_i's, define $s(\mathcal{E}):=\frac{1}{2}\varphi(wt(\mathcal{E}))$, $r(\mathcal{E}):=\frac{1}{2}r(\varphi(wt(\mathcal{E})))$, $\overline{r}(\mathcal{E}):=\frac{1}{2}\overline{r}(\varphi(wt(\mathcal{E})))$, where $wt(\mathcal{E})$ denotes the weight of $\mathcal{E}$ (E_i having weight α_i), and similarly for every monomial $\mathcal{F}$ in the F_i's, (F_i having weight $-\alpha_i$). Then the values of DRT pairings on PBW monomials are given by

$$\begin{aligned}&\pi_-^\varphi\left(\prod_{r=N}^1 F_{\alpha^r}^{f_r}\cdot L_\lambda,\prod_{r=N}^1 E_{\alpha^r}^{e_r}\cdot K_\alpha\right)=\\&\qquad=q^{-\left(r(\lambda)-r\left(\prod_{r=N}^1 F_{\alpha^r}^{f_r}\right)\middle|\alpha-s\left(\prod_{r=N}^1 E_{\alpha^r}^{e_r}\right)\right)}\prod_{r=1}^N\delta_{e_r,f_r}\frac{[e_r]_{q_{\alpha^r}}!\;q_{\alpha^r}^{+\binom{e_r}{2}}}{\left(q_{\alpha^r}^{-1}-q_{\alpha^r}\right)^{e_r}}\\&\pi_+^\varphi\left(\prod_{r=N}^1 F_{\alpha^r}^{f_r}\cdot K_\alpha,\prod_{r=N}^1 E_{\alpha^r}^{e_r}\cdot L_\lambda\right)=\\&\qquad=q^{-\left(r(\alpha)-r\left(\prod_{r=N}^1 F_{\alpha^r}^{f_r}\right)\middle|\lambda-s\left(\prod_{r=N}^1 E_{\alpha^r}^{e_r}\right)\right)}\prod_{r=1}^N\delta_{e_r,f_r}\frac{[e_r]_{q_{\alpha^r}}!\;q_{\alpha^r}^{+\binom{e_r}{2}}}{\left(q_{\alpha^r}^{-1}-q_{\alpha^r}\right)^{e_r}}\end{aligned}\tag{2.2-a}$$

$$\overline{\pi_-^{\varphi}}\left(\prod_{r=N}^{1} L_\lambda \cdot E_{\alpha^r}^{e_r}, K_\alpha \cdot \prod_{r=N}^{1} F_{\alpha^r}^{f_r}\right) = \\ = q^{\left(\overline{r}(\alpha)-\overline{r}\left(\prod_{r=N}^{1} F_{\alpha^r}^{f_r}\right)\middle|\lambda-s\left(\prod_{r=N}^{1} E_{\alpha^r}^{e_r}\right)\right)} \prod_{r=1}^{N} \delta_{e_r,f_r} \frac{[e_r]_{q_{\alpha^r}}! \; q_{\alpha^r}^{-\binom{e_r}{2}}}{\left(q_{\alpha^r}-q_{\alpha^r}^{-1}\right)^{e_r}}$$

$$\overline{\pi_+^{\varphi}}\left(\prod_{r=N}^{1} K_\alpha \cdot E_{\alpha^r}^{e_r}, L_\lambda \cdot \prod_{r=N}^{1} F_{\alpha^r}^{f_r}\right) = \\ = q^{\left(r(\lambda)-r\left(\prod_{r=N}^{1} F_{\alpha^r}^{f_r}\right)\middle|\alpha-s\left(\prod_{r=N}^{1} E_{\alpha^r}^{e_r}\right)\right)} \prod_{r=1}^{N} \delta_{e_r,f_r} \frac{[e_r]_{q_{\alpha^r}}! \; q_{\alpha^r}^{-\binom{e_r}{2}}}{\left(q_{\alpha^r}-q_{\alpha^r}^{-1}\right)^{e_r}} \qquad (2.2-b)$$

(cf. [CV-2] §1, taking care of our different normalizations; see also [DD] and [DL]).

2.5 Integer forms and duality. Let $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{b}_-)$ be the $k[q,q^{-1}]$-subalgebra of $U_{q,\varphi}^{Q}(\mathfrak{b}_-)$ generated by

$$\left\{ F_i^{(m)}, \binom{K_i;c}{t}, K_i^{-1} \,\middle|\, m,c,t \in \mathbb{N}; i=1,\dots,n \right\}$$

where $F_i^{(m)} := F_i^m / [m]_{q_i}!$ and $\binom{K_i;c}{t} := \prod_{s=1}^{t} \frac{K_i q_i^{c-s+1}-1}{q_i^s-1}$ are the so-called "divided powers"; it is known (cf. [Lu-2], [DL], [CV-2]) that $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{b}_-)$ is a Hopf subalgebra of $U_{q,\varphi}^{Q}(\mathfrak{b}_-)$, having a PBW basis (as a $k[q,q^{-1}]$-module) of increasing ordered monomials

$$\left\{ \prod_{i=1}^{n} \binom{K_i;0}{t_i} K_i^{-Ent(t_i/2)} \cdot \prod_{r=1}^{N} F_{\alpha^r}^{(n_r)} \,\middle|\, t_1,\dots,t_n,n_1,\dots,n_N \in \mathbb{N} \right\}$$

and a similar PBW basis of decreasing ordered monomials; in particular $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{b}_-)$ is a $k[q,q^{-1}]$-form of $U_{q,\varphi}^{Q}(\mathfrak{b}_-)$. A similar definition gives us the $k[q,q^{-1}]$-subalgebra $\widehat{U}_{q,\varphi}^{P}(\mathfrak{b}_+)$ of $U_{q,\varphi}^{P}(\mathfrak{b}_+)$ generated by divided powers in the E_i's and L_i's, which is an integer form of $U_{q,\varphi}^{P}(\mathfrak{b}_+)$ with a PBW basis of decreasing ordered monomials and a PBW basis of increasing ordered monomials. With the same procedure one defines Hopf algebras $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{b}_+)$ and $\widehat{U}_{q,\varphi}^{P}(\mathfrak{b}_-)$ (over $k[q,q^{-1}]$) and locates PBW bases for them.

Now let $\widetilde{U}_{q,\varphi}^{P}(\mathfrak{b}_+)$ be the $k[q,q^{-1}]$-subalgebra of $U_{q,\varphi}^{P}(\mathfrak{b}_+)$ generated by

$$\{\overline{E}_{\alpha^1},\dots,\overline{E}_{\alpha^N}\} \cup \{L_1^{\pm},\dots,L_n^{\pm}\}$$

where $\overline{E}_{\alpha^r} := (q_{\alpha^r}-q_{\alpha^r}^{-1})E_{\alpha^r}$, $\forall\, r=1,\dots,N$; then (cf. [DKP], [DP]) $\widetilde{U}_{q,\varphi}^{P}(\mathfrak{b}_+)$ is a Hopf subalgebra of $U_{q,\varphi}^{P}(\mathfrak{b}_+)$, having a PBW basis (as a $k[q,q^{-1}]$-module)

$$\left\{ \prod_{i=1}^{n} L_i^{t_i} \cdot \prod_{r=1}^{N} \overline{E}_{\alpha^r}^{n_r} \,\middle|\, t_1,\dots,t_n \in \mathbb{Z}; n_1,\dots,n_N \in \mathbb{N} \right\}$$

of increasing ordered monomials and a similar PBW basis of decreasing ordered monomials; in particular $\widetilde{U}_{q,\varphi}^P(\mathfrak{b}_+)$ is a $k[q,q^{-1}]$-form of $U_{q,\varphi}^P(\mathfrak{b}_+)$. Similarly we define $\widetilde{U}_{q,\varphi}^Q(\mathfrak{b}_-)$ to be the $k[q,q^{-1}]$-subalgebra of $U_{q,\varphi}^Q(\mathfrak{b}_-)$ generated by $\{\overline{F}_{\alpha^1},\dots,\overline{F}_{\alpha^N}\}\cup\{K_1^\pm,\dots,K_n^\pm\}$ (with $\overline{F}_{\alpha^r} := (q_{\alpha^r}-q_{\alpha^r}^{-1})F_{\alpha^r}$, $\forall\, r=1,\dots,N$), which is an integer form of $U_{q,\varphi}^Q(\mathfrak{b}_-)$, having PBW $k[q,q^{-1}]$-bases of decreasing or increasing ordered monomials. With the same procedure one defines Hopf algebras $\widetilde{U}_{q,\varphi}^Q(\mathfrak{b}_+)$ and $\widetilde{U}_{q,\varphi}^P(\mathfrak{b}_-)$ (over $k[q,q^{-1}]$) and locates PBW bases for them.

Similar constructions (and corresponding notations) and results also hold for the algebras $U_{q,\varphi}^M(\mathfrak{t})$, $U_{q,\varphi}(\mathfrak{n}_+)$, and $U_{q,\varphi}(\mathfrak{n}_-)$.

Finally, from the very definitions and from (2.2) one immediately gets (as in [DL] §3)

$$\begin{aligned}
\widehat{U}_{q,\varphi}^Q(\mathfrak{t}) &= \{\, y\in U_{q,\varphi}^Q(\mathfrak{t}) \mid \pi_-^\varphi(\widetilde{U}_{q,\varphi}^P(\mathfrak{t}),y) \leq k[q,q^{-1}]\,\} = \\
&= \{\, x\in U_{q,\varphi}^Q(\mathfrak{t}) \mid \pi_+^\varphi(x,\widetilde{U}_{q,\varphi}^P(\mathfrak{t})) \leq k[q,q^{-1}]\,\} \\
\widetilde{U}_{q,\varphi}^P(\mathfrak{t}) &= \{\, y\in U_{q,\varphi}^P(\mathfrak{t}) \mid \pi_+^\varphi(\widehat{U}_{q,\varphi}^Q(\mathfrak{t}),y) \leq k[q,q^{-1}]\,\} = \\
&= \{\, x\in U_{q,\varphi}^P(\mathfrak{t}) \mid \pi_-^\varphi(x,\widehat{U}_{q,\varphi}^Q(\mathfrak{t})) \leq k[q,q^{-1}]\,\} \\
\widehat{U}_{q,\varphi}^Q(\mathfrak{b}_-)_{op} &= \{\, x\in U_{q,\varphi}^Q(\mathfrak{b}_-)_{\mathrm{op}} \mid \pi_+^\varphi(x,\widetilde{U}_{q,\varphi}^P(\mathfrak{b}_+)) \leq k[q,q^{-1}]\,\} \\
\widetilde{U}_{q,\varphi}^P(\mathfrak{b}_-)_{op} &= \{\, x\in U_{q,\varphi}^P(\mathfrak{b}_-)_{op} \mid \pi_-^\varphi(x,\widehat{U}_{q,\varphi}^Q(\mathfrak{b}_+)) \leq k[q,q^{-1}]\,\} \\
\widehat{U}_{q,\varphi}^Q(\mathfrak{b}_+)_{\mathrm{op}} &= \{\, x\in U_{q,\varphi}^Q(\mathfrak{b}_+)_{op} \mid \overline{\pi_+^\varphi}(x,\widetilde{U}_{q,\varphi}^P(\mathfrak{b}_-)) \leq k[q,q^{-1}]\,\} \\
\widetilde{U}_{q,\varphi}^P(\mathfrak{b}_+)_{op} &= \{\, x\in U_{q,\varphi}^P(\mathfrak{b}_+)_{op} \mid \overline{\pi_-^\varphi}(x,\widehat{U}_{q,\varphi}^Q(\mathfrak{b}_-)) \leq k[q,q^{-1}]\,\} \\
\widetilde{U}_{q,\varphi}^P(\mathfrak{b}_+) &= \{\, y\in U_{q,\varphi}^P(\mathfrak{b}_+) \mid \pi_+^\varphi(\widehat{U}_{q,\varphi}^Q(\mathfrak{b}_-)_{op},y) \leq k[q,q^{-1}]\,\} \\
\widehat{U}_{q,\varphi}^Q(\mathfrak{b}_+) &= \{\, y\in U_{q,\varphi}^Q(\mathfrak{b}_+) \mid \pi_-^\varphi(\widetilde{U}_{q,\varphi}^P(\mathfrak{b}_-)_{op},y) \leq k[q,q^{-1}]\,\} \\
\widetilde{U}_{q,\varphi}^P(\mathfrak{b}_-) &= \{\, y\in U_{q,\varphi}^P(\mathfrak{b}_-) \mid \overline{\pi_+^\varphi}(\widehat{U}_{q,\varphi}^Q(\mathfrak{b}_+)_{op},y) \leq k[q,q^{-1}]\,\} \\
\widehat{U}_{q,\varphi}^Q(\mathfrak{b}_-) &= \{\, y\in U_{q,\varphi}^Q(\mathfrak{b}_-) \mid \overline{\pi_-^\varphi}(\widetilde{U}_{q,\varphi}^P(\mathfrak{b}_+)_{op},y) \leq k[q,q^{-1}]\,\}.
\end{aligned} \tag{2.3}$$

§ 3 Quantization of $U(\mathfrak{g}^\tau)$

3.1 Drinfeld's double. Let H_-, H_+ be two arbitrary Hopf algebras on the ground field (or ring) F, and let $\pi\colon (H_-)_{op}\otimes H_+ \to F$ be any arbitrary Hopf pairing. The *Drinfeld's double* $D=D(H_-,H_+,\pi)$ is the algebra $T(H_-\oplus H_+)\big/\mathcal{R}$, where $\mathcal{R}$ is the ideal of relations

$$1_{H_-} = 1 = 1_{H_+} \quad , \qquad x\otimes y = xy \qquad\qquad \forall\, x,y\in H_+ \ \text{ or } \ x,y\in H_-$$
$$\sum_{(x),(y)} \pi(y_{(2)},x_{(2)})\, x_{(1)}\otimes y_{(1)} = \sum_{(x),(y)} \pi(y_{(1)},x_{(1)})\, y_{(2)}\otimes x_{(2)} \qquad \forall\, x\in H_+,\, y\in H_-\,.$$

Then (cf. [DL], Theorem 3.6) D has a canonical structure of Hopf algebra such that H_-, H_+ are Hopf subalgebras of it and multiplication yields isomorphisms of coalgebras

$$H_+ \otimes H_- \hookrightarrow D \otimes D \xrightarrow{m} D\,, \quad H_- \otimes H_+ \hookrightarrow D \otimes D \xrightarrow{m} D\,. \tag{3.1}$$

We apply this to get the Drinfeld's double $D_{q,\varphi}^M(\mathfrak{g}) := D(U_{q,\varphi}^Q(\mathfrak{b}_-), U_{q,\varphi}^M(\mathfrak{b}_+), \pi_+^\varphi)$ (for any lattice M, with $Q \leq M \leq P$) which we call *quantum double*; from the very definition, $D_{q,\varphi}^M(\mathfrak{g})$ is generated by K_α, L_μ, F_i, E_i — identified with $1 \otimes K_\alpha$, $L_\mu \otimes 1$, $1 \otimes F_i$, $E_i \otimes 1$ when thinking at $D_{q,\varphi}^M(\mathfrak{g}) \cong U_{q,\varphi}^M(\mathfrak{b}_+) \otimes U_{q,\varphi}^Q(\mathfrak{b}_-)$ — ($\alpha \in Q$, $\mu \in M$, $i = 1, \dots, n$), while relations defining the ideal $\mathcal{R}$ clearly reduce to commutation relations between generators, namely (for all $i, j = 1, \dots, n$)

$$\begin{aligned} K_\alpha L_\mu = L_\mu K_\alpha\,, \qquad K_\alpha E_j =& q_i^{(\alpha|\alpha_j)} E_j K_\alpha\,, \qquad L_\mu F_j = q_i^{-(\mu|\alpha_j)} F_j L_\mu\,, \\ E_i F_j - F_j E_i =& \delta_{ij} \frac{L_{\alpha_i} - K_{-\alpha_i}}{q_i - q_i^{-1}}\,. \end{aligned} \tag{3.2}$$

For later use we also record the following (deduced from (3.2))

$$\begin{aligned} E_i^r F_i^s &= \sum_{t \geq 0}^{t \leq r,s} \begin{bmatrix} r \\ t \end{bmatrix}_{q_i} \begin{bmatrix} s \\ t \end{bmatrix}_{q_i} [t]_{q_i}!^2 \cdot F_i^{s-t} \cdot \begin{bmatrix} K_{\alpha_i} \overset{\otimes}{\otimes}; 2t-r-s \\ t \end{bmatrix} \cdot E_i^{r-t} \\ F_i^s E_i^r &= \sum_{t \geq 0}^{t \leq r,s} \begin{bmatrix} r \\ t \end{bmatrix}_{q_i} \begin{bmatrix} s \\ t \end{bmatrix}_{q_i} [t]_{q_i}!^2 \cdot \cdot E_i^{r-t} \cdot \begin{bmatrix} K_{-\alpha_i} \overset{\otimes}{\otimes}; 2t-r-s \\ t \end{bmatrix} \cdot F_i^{s-t} \end{aligned} \tag{3.3}$$

($\forall\, i = 1, \dots, n$) where $\begin{bmatrix} K_{\alpha_i} \overset{\otimes}{\otimes}; c \\ t \end{bmatrix} := \prod_{p=1}^t \frac{q_i^{c-p+1} \cdot L_{\alpha_i} - K_{-\alpha_i} \cdot q_i^{-c+p-1}}{q_i^p - q_i^{-p}}$ and $\begin{bmatrix} K_{-\alpha_i} \overset{\otimes}{\otimes}; c \\ t \end{bmatrix} :=$ $:= \prod_{p=1}^t \frac{q_i^{c-p+1} \cdot K_{-\alpha_i} - L_{\alpha_i} \cdot q_i^{-c+p-1}}{q_i^p - q_i^{-p}}$ for all $c \in \mathbb{Z}$, $t \in \mathbb{N}$.

Finally, PBW bases of quantum Borel algebras clearly provide (tensor) PBW bases of $D_{q,\varphi}^M(\mathfrak{g})$ (identified with $U_{q,\varphi}^M(\mathfrak{b}_+) \otimes U_{q,\varphi}^Q(\mathfrak{b}_-)$, as we shall always do in the sequel).

3.3 The quantum algebra $U_{q,\varphi}^M(\mathfrak{g})$. Let $\mathfrak{K}_\varphi^P$ be the ideal of $D_{q,\varphi}^P(\mathfrak{g})$ generated by the elements $K \otimes 1 - 1 \otimes K$, $K \in U_q^Q(\mathfrak{t})$; $\mathfrak{K}_\varphi^P$ is in fact a Hopf ideal, whence $D_{q,\varphi}^P(\mathfrak{g}) \big/ \mathfrak{K}_\varphi^P$ is a Hopf algebra; then the above presentation of $D_{q,\varphi}^P(\mathfrak{g})$ yields the following one of $U_{q,\varphi}^P(\mathfrak{g}) := D_{q,\varphi}^P(\mathfrak{g}) \big/ \mathfrak{K}_\varphi^P$: it is the associative $k(q)$-algebra with 1 given by generators

$$F_i\,,\ L_\lambda \ (\text{or } K_\lambda)\,,\ E_i \quad (\lambda \in P; i = 1, \dots, n)$$

and relations

$$\begin{gathered} L_0 = 1\,,\ L_\lambda L_\mu = L_{\lambda+\mu} = L_\mu L_\lambda\,,\ L_\lambda F_i = q^{-(\alpha_j|\lambda)} F_i L_\lambda\,,\ L_\lambda E_i = q^{(\alpha_j|\lambda)} E_i L_\lambda \\ E_i F_h - F_h E_i = \delta_{ih} \frac{L_{\alpha_i} - L_{-\alpha_i}}{q_i - q_i^{-1}} \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0,\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0 \end{gathered} \tag{3.4}$$

(for all $\lambda \in P$, $i, j, h = 1, \dots, n$ with $i \neq j$) with the Hopf structure given by

$$\begin{aligned} \Delta_\varphi(F_i) = F_i \otimes L_{-\alpha_i-\tau_i} + L_{\tau_i} \otimes F_i \,, \quad & \varepsilon_\varphi(F_i) = 0 \,, \quad S_\varphi(F_i) = -F_i L_{\alpha_i} \\ \Delta_\varphi(L_\lambda) = L_\lambda \otimes L_\lambda \,, \quad & \varepsilon_\varphi(L_\lambda) = 1 \,, \quad S_\varphi(L_\lambda) = L_{-\lambda} \\ \Delta_\varphi(E_i) = E_i \otimes L_{\tau_i} + L_{\alpha_i-\tau_i} \otimes F_i \,, \quad & \varepsilon_\varphi(E_i) = 0 \,, \quad S_\varphi(F_i) = -L_{-\alpha_i} E_i \end{aligned} \tag{3.5}$$

for all $i = 1, \dots, n$, $\lambda \in P$. Similarly, with any lattice M ($Q \leq M \leq P$) we define $U_{q,\varphi}^M(\mathfrak{g}) := D_{q,\varphi}^M(\mathfrak{g}) \Big/ \mathfrak{K}_\varphi^M$ which has a presentation by generators and relations as in (10.4–5), with M instead of P; in particular we shall study $U_{q,\varphi}^Q(\mathfrak{g})$.[3]

Finally we shall denote with $pr_\varphi^M \colon D_{q,\varphi}^M(\mathfrak{g}) \twoheadrightarrow D_{q,\varphi}^M(\mathfrak{g}) \Big/ \mathfrak{K}_\varphi^M = U_{q,\varphi}^M(\mathfrak{g})$ the canonical Hopf algebra epimorphism, and we shall use notation $K_\alpha := L_\alpha \ \forall \, \alpha \in Q$.

3.4 Integer forms of $U_{q,\varphi}^M(\mathfrak{g})$. Let $\widehat{U}_{q,\varphi}^Q(\mathfrak{g})$ be the $k[q,q^{-1}]$-subalgebra of $U_{q,\varphi}^Q(\mathfrak{g})$ generated by

$$\left\{ F_i^{(\ell)}, \begin{pmatrix} K_i; c \\ t \end{pmatrix}, K_i^{-1}, E_i^{(m)} \,\middle|\, \ell, c, t, m \in \mathbb{N}; i = 1, \dots, n \right\} \; ;$$

then (cf. [DL], §3) this is a Hopf subalgebra of $U_{q,\varphi}^Q(\mathfrak{g})$, with PBW basis (over $k[q,q^{-1}]$)

$$\left\{ \prod_{r=N}^{1} E_{\alpha^r}^{(n_r)} \cdot \prod_{i=1}^{n} \begin{pmatrix} K_i; 0 \\ t_i \end{pmatrix} K_i^{-Ent(t_i/2)} \cdot \prod_{r=1}^{N} F_{\alpha^r}^{(m_r)} \,\middle|\, n_1, \dots, n_N, t_1, \dots, t_n, m_1, \dots, m_N \in \mathbb{N} \right\};$$

this is also a $k(q)$-basis of $U_{q,\varphi}^Q(\mathfrak{g})$, hence clearly $\widehat{U}_q^Q(\mathfrak{g})$ is a $k[q,q^{-1}]$-form of $U_{q,\varphi}^Q(\mathfrak{g})$.

Let $\widetilde{U}_{q,\varphi}^P(\mathfrak{g})$ be the $k[q,q^{-1}]$-subalgebra of $U_{q,\varphi}^P(\mathfrak{g})$ generated by

$$\{\overline{F}_{\alpha^1}, \dots, \overline{F}_{\alpha^N}\} \cup \{ L_1^{\pm}, \dots, L_n^{\pm} \} \cup \{\overline{F}_{\alpha^1}, \dots, \overline{E}_{\alpha^N}\}$$

(cf. [DP], §12; see also [DK] and [DKP]); then this is a Hopf subalgebra of $U_{q,\varphi}^P(\mathfrak{g})$, having a PBW basis (as a $k[q,q^{-1}]$-module)

$$\left\{ \prod_{r=N}^{1} \overline{E}_{\alpha^r}^{n_r} \cdot \prod_{i=1}^{n} L_i^{t_i} \cdot \prod_{r=1}^{N} \overline{F}_{\alpha^r}^{m_r} \,\middle|\, t_1, \dots, t_n \in \mathbb{Z}; n_1, \dots, n_N, m_1, \dots, m_N \in \mathbb{N} \right\};$$

this is also a $k(q)$-basis of $U_{q,\varphi}^P(\mathfrak{g})$, hence clearly $\widetilde{U}_{q,\varphi}^P(\mathfrak{g})$ is a $k[q,q^{-1}]$-form of $U_{q,\varphi}^P(\mathfrak{g})$.

3.5 Specialization at roots of 1 and quantum Frobenius morphisms. Since the integer forms we defined are modules over $k[q,q^{-1}]$, we can specialize them at special values of q. To this end, we assume for simplicity that our ground field k contains all roots

[3] Notice that for $\varphi = 0$ one recovers the usual 1-parameter quantum enveloping algebras: thus $U_{q,0}^M(\mathfrak{g})$ is nothing but $U_{q,M}(\mathfrak{g})$ of [DP], §9, hence in particular for $M = Q$ we recover the "quantum group of adjoint type" of Jimbo, Lustzig, and others.

of unity (so that $k[q,q^{-1}]\Big/(q-\varepsilon)=k$ for any root of unity ε; otherwise, specialization will extend the ground field to $k[q,q^{-1}]\Big/p_\ell(q)=k(\varepsilon)$, where $p_\ell(q)$ is the ℓ-th cyclotomic polynomial and ε denotes a primitive ℓ-th root of 1); notice that this is the case for $k=\mathbb{C}$.

Let us begin with $\widehat{U}_{q,\varphi}^Q(\mathfrak{g})$. For the case $\varepsilon=1$, set

$$\widehat{U}_{1,\varphi}^Q(\mathfrak{g}) := \widehat{U}_{q,\varphi}^Q(\mathfrak{g})\Big/(q-1)\widehat{U}_q^Q(\mathfrak{g}) \,\cong\, \widehat{U}_{q,\varphi}^Q(\mathfrak{g})\otimes_{k[q,q^{-1}]} k$$

(where k is meant as a $k[q,q^{-1}]$-algebra via $k\cong k[q,q^{-1}]/(q-1)$); let $p_1 \colon \widehat{U}_{q,\varphi}^Q(\mathfrak{g}) \twoheadrightarrow \widehat{U}_{1,\varphi}^Q(\mathfrak{g})$ be the canonical projection and set $f_i := p_1\Big(F_i^{(1)}\Big)$, $h_i := p_1\Big(\Big(\begin{smallmatrix} K_i;0 \\ 1 \end{smallmatrix}\Big)\Big)$, $e_i := p_1\Big(E_i^{(1)}\Big)$, for all $i=1,\dots,n$. Furthermore, observe that the algebras $\widehat{U}_{q,\varphi}^Q(\mathfrak{g})$ for various φ are all canonically isomorphic; in particular they have the same presentation (*as algebras*) by generators and relations as $\widehat{U}_q^Q(\mathfrak{g})=\widehat{U}_{q,0}^Q(\mathfrak{g})$, which is given in [DL], §3.4; then one immediately finds that $\widehat{U}_{1,\varphi}^Q(\mathfrak{g})$ is a Poisson Hopf coalgebra, with a presentation by generators (f_i,h_i,e_i) and relations which is exactly the same of $U(\mathfrak{g}^\tau)$ (with τ given by (2.1)) given in §1, hence we have a Poisson Hopf coalgebra isomorphism

$$\widehat{U}_{1,\varphi}^Q(\mathfrak{g})\cong U(\mathfrak{g}^\tau)\;; \tag{3.6}$$

in a word, $\widehat{U}_{q,\varphi}^Q(\mathfrak{g})$ specializes to $U(\mathfrak{g}^\tau)$ for $q\to 1$: in symbols, $\widehat{U}_{q,\varphi}^Q(\mathfrak{g}) \xrightarrow{q\to 1} U(\mathfrak{g}^\tau)$.

Let now ε be a primitive ℓ-th root of 1, for ℓ *odd*, $\ell>d:=\max_i\{d_i\}_i$, and set

$$\widehat{U}_{\varepsilon,\varphi}^Q(\mathfrak{g}) := \widehat{U}_{q,\varphi}^Q(\mathfrak{g})\Big/(q-\varepsilon)\widehat{U}_{q,\varphi}^Q(\mathfrak{g}) \,\cong\, \widehat{U}_{q,\varphi}^Q(\mathfrak{g})\otimes_{k[q,q^{-1}]} k$$

(where k is meant as a $k[q,q^{-1}]$-algebra via $k\cong k[q,q^{-1}]\Big/(q-\varepsilon)$); from [CV-2], §3.2 (cf. also [Lu-2], Theorem 8.10, and [DL], Theorem 6.3, for the one-parameter case) we record the existence of a special Hopf algebra epimorphism

$$\widehat{\mathcal{F}r}_{\mathfrak{g}^\tau}\colon \widehat{U}_{\varepsilon,\varphi}^Q(\mathfrak{g}) \twoheadrightarrow \widehat{U}_{1,\varphi}^Q(\mathfrak{g})\cong U(\mathfrak{g}^\tau) \tag{3.7}$$

When $\varphi=0$ — whence $\tau=0$ — and $\ell=p$ is prime, it is shown in [Lu-2], §8.15, that $\widehat{\mathcal{F}r}_{\mathfrak{g}^0}$ can be regarded as a *lifting of the Frobenius morphism* $G_{\mathbb{Z}_p}\to G_{\mathbb{Z}_p}$ *to characteristic zero*; for this reason, we shall refer to $\widehat{\mathcal{F}r}_{\mathfrak{g}^\tau}$ as a *quantum Frobenius morphism.*

Now for $\widetilde{U}_{q,\varphi}^P(\mathfrak{g})$. Set

$$\widetilde{U}_{1,\varphi}^P(\mathfrak{g}) := \widetilde{U}_{q,\varphi}^P(\mathfrak{g})\Big/(q-1)\widetilde{U}_{q,\varphi}^P(\mathfrak{g}) \,\cong\, \widetilde{U}_{q,\varphi}^P(\mathfrak{g})\otimes_{k[q,q^{-1}]} k\;;$$

it is known (cf. [DP], Theorem 12.1, and [DKP], Theorem 7.6 and Remark 7.7 *(c)*) that

$$\widetilde{U}_{1,\varphi}^P(\mathfrak{g})\cong F[H^\tau] \tag{3.8}$$

as Poisson Hopf algebras over k (here H^τ is the Poisson group defined in §1.5); in a word, $\widetilde{U}_{q,\varphi}^P(\mathfrak{g})$ specializes to $F[H^\tau]$ as $q \to 1$, or $\widetilde{U}_{q,\varphi}^P(\mathfrak{g}) \xrightarrow{q\to 1} F[H^\tau]$.

For ε a primitive ℓ-th root of 1 (ℓ *odd*, $\ell > d := \max_i\{d_i\}$), we set

$$\widetilde{U}_{\varepsilon,\varphi}^P(\mathfrak{g}) := \widetilde{U}_{q,\varphi}^P(\mathfrak{g}) \Big/ (q-\varepsilon)\widetilde{U}_{q,\varphi}^P(\mathfrak{g}) \cong \widetilde{U}_{q,\varphi}^P(\mathfrak{g}) \otimes_{k[q,q^{-1}]} k\;;$$

from [DKP], §§7.6–7 (cf. also [DP] for the one-parameter case) we record the existence of a special Hopf algebra monomorphism

$$\widetilde{\mathcal{F}r}_{\mathfrak{g}^\tau} \colon F[H^\tau] \cong \widetilde{U}_{1,\varphi}^P(\mathfrak{g}) \hookrightarrow \widetilde{U}_{\varepsilon,\varphi}^P(\mathfrak{g}) \tag{3.9}$$

Just as for $\widehat{\mathcal{F}r}_{\mathfrak{g}^\tau}$, we shall refer to $\widetilde{\mathcal{F}r}_{\mathfrak{g}^\tau}$ as a *quantum Frobenius morphism*; in fact in case $\varphi = 0$ and $\ell = p$ prime it can be regarded as a *lifting of the Frobenius morphism* $H_{\mathbb{Z}_p} \to H_{\mathbb{Z}_p}$ *to characteristic zero*[4].

§ 4 Quantum function algebras

4.1 The quantum function algebras $F_{q,\varphi}^M[B_\pm]$ (cf. [DL] §4.1, [CV-2] §§2-3). Let $F_{q,\varphi}^M[B_\pm]$ be the *quantum function algebra* relative to $U_{q,\varphi}^{M'}(\mathfrak{b}_\pm)$, defined as the algebra of matrix coefficients of *positive*[5] finite dimensional representations of $U_{q,\varphi}^{M'}(\mathfrak{b}_\pm)$ (cf. [DL], §4). Then $F_{q,\varphi}^M[B_\pm]$ is a Hopf algebra, which we call dual of $U_{q,\varphi}^{M'}(\mathfrak{b}_\pm)$ because there exists a natural perfect Hopf pairing (evaluation) among them; it is also possible to realize $F_{q,\varphi}^M[B_\pm]$ as a Hopf subalgebra of $U_{q,\varphi}^{M'}(\mathfrak{b}_\pm)^\circ$ (hereafter for any Hopf algebra H we will denote with H° ($\subseteq H^*$) its dual Hopf algebra, in the standard sense of [SW], ch. VI). Notice that, since multiparameter quantum enveloping algebras (with the same weight lattice M) are all isomorphic *as algebras*, then multiparameter quantum function algebras are all isomorphic *as coalgebras*. DRT pairings provide Hopf algebra isomorphisms

$$F_{q,\varphi}^P[B_+] \cong U_{q,\varphi}^P(\mathfrak{b}_-)_{op}\,, \qquad F_{q,\varphi}^Q[B_+] \cong U_{q,\varphi}^Q(\mathfrak{b}_-)_{op}\,, \qquad F_{q,\varphi}^P[B_-] \cong U_{q,\varphi}^P(\mathfrak{b}_+)_{op} \tag{4.1}$$

induced by the pairing π_-^φ, resp. π_+^φ, resp. $\overline{\pi_-^\varphi}$ (cf. [DL], Proposition 4.2, and [CV-2], §2.3); by means of these, DRT pairings can be seen as natural evaluation pairings.

4.2 Integer forms of $F_{q,\varphi}^M[B_\pm]$. In §2.5 we introduced integer forms of $U_{q,\varphi}^{M'}(\mathfrak{b}_\pm)$: here we define the corresponding function algebras, which will result to be integer forms of $F_{q,\varphi}^M[B_\pm]$. We set

$$\begin{aligned} \widehat{F}_{q,\varphi}^P[B_\pm] &:= \Big\{ f \in F_{q,\varphi}^P[B_\pm] \,\Big|\, \Big\langle f, \widehat{U}_{q,\varphi}^Q(\mathfrak{b}_\pm) \Big\rangle \subseteq k[q,q^{-1}] \Big\} \\ \widetilde{F}_{q,\varphi}^Q[B_\pm] &:= \Big\{ f \in F_{q,\varphi}^Q[B_\pm] \,\Big|\, \Big\langle f, \widetilde{U}_{q,\varphi}^P(\mathfrak{b}_\pm) \Big\rangle \subseteq k[q,q^{-1}] \Big\} \end{aligned} \tag{4.2}$$

[4]Here $H_{\mathbb{Z}_p}$ denotes the Chevalley-type group-scheme over $\mathbb{Z}_p$ that one can clearly build up from the "Cartan datum" associated to H (mimicking the usual procedure one follows for $G_{\mathbb{Z}_p}$).

[5]i. e. those for which there exists a basis in which the operators K_μ ($\mu \in M'$) act diagonally with eigenvalues powers of q.

where $\langle\ ,\ \rangle\colon F_{q,\varphi}^M[B_\pm] \otimes U_{q,\varphi}^{M'}(\mathfrak{b}_\pm) \to k(q)$ is the natural evaluation pairing; then from (2.2) and (4.1) one gets

$$\widehat{F}_{q,\varphi}^P[B_\pm] \cong \widetilde{U}_{q,\varphi}^P(\mathfrak{b}_\mp)_{op}\,, \quad \widetilde{F}_{q,\varphi}^Q[B_\pm] \cong \widehat{U}_{q,\varphi}^Q(\mathfrak{b}_\mp)_{op}\,; \tag{4.3}$$

in particular $\widehat{F}_{q,\varphi}^P[B_\pm]$ and $\widetilde{F}_{q,\varphi}^Q[B_\pm]$ are Hopf $k[q,q^{-1}]$-subalgebras and integer forms (over $k[q,q^{-1}]$) of $F_{q,\varphi}^P[B_\pm]$ and $F_{q,\varphi}^P[B_\pm]$ respectively; we also have

$$\widehat{F}_{q,\varphi}^P[B_\pm] \cong \widetilde{U}_{q,\varphi}^P(\mathfrak{b}_\mp)^{op}\,, \quad \widetilde{F}_{q,\varphi}^Q[B_\pm] \cong \widehat{U}_{q,\varphi}^Q(\mathfrak{b}_\mp)^{op}\,, \tag{4.4}$$

different isomorphisms arising from different kinds of DRT pairing (cf. also (2.3)).

4.3 The quantum function algebra $F_{q,\varphi}^M[G]$ and its integer forms. Like in §4.1, we let $F_{q,\varphi}^M[G]$ be the *quantum function algebra* relative to $U_{q,\varphi}^{M'}(\mathfrak{g})$, defined as the algebra of matrix coefficients of *positive* finite dimensional representations of $U_{q,\varphi}^{M'}(\mathfrak{g})$ (cf. [DL], §4, and [CV-2], §2.1); this is a Hopf algebra, which we call dual of $U_{q,\varphi}^{M'}(\mathfrak{g})$ because there exists a natural perfect Hopf pairing (evaluation) among them; in fact $F_{q,\varphi}^M[G]$ is a Hopf subalgebra of $U_{q,\varphi}^{M'}(\mathfrak{g})^\circ$.

As for integer forms, let

$$\begin{aligned} \widehat{F}_{q,\varphi}^P[G] &:= \left\{ f \in F_{q,\varphi}^P[G] \,\middle|\, \left\langle f, \widehat{U}_{q,\varphi}^Q(\mathfrak{g}) \right\rangle \subseteq k[q,q^{-1}] \right\} \\ \widetilde{F}_{q,\varphi}^Q[G] &:= \left\{ f \in F_{q,\varphi}^Q[G] \,\middle|\, \left\langle f, \widetilde{U}_{q,\varphi}^P(\mathfrak{g}) \right\rangle \subseteq k[q,q^{-1}] \right\} \end{aligned} \tag{4.5}$$

where $\langle\ ,\ \rangle\colon F_{q,\varphi}^M[G] \otimes U_{q,\varphi}^{M'}(\mathfrak{g}) \to k(q)$ is the natural evaluation pairing; these are Hopf subalgebras (over $k[q,q^{-1}]$) of $F_{q,\varphi}^P[G]$ and $F_{q,\varphi}^Q[G]$.

4.4 Specialization at roots of 1. It is shown in [CV-2] (generalizing [DL]) that

$$\widehat{F}_{q,\varphi}^P[G] \xrightarrow{\ q\to 1\ } F[G^\tau] \tag{4.6}$$

i. e. $\widehat{F}_{1,\varphi}^P[G] := \widehat{F}_{q,\varphi}^P[G] \Big/ (q-1)\widehat{F}_{q,\varphi}^P[G] \cong F[G^\tau]$ as Poisson Hopf k-algebras; in fact this result arises as dual of $\widehat{U}_{q,\varphi}^Q(\mathfrak{g}) \xrightarrow{\ q\to 1\ } U(\mathfrak{g})$. For ε a primitive ℓ-th root of 1 (ℓ *odd*, $\ell > d := \max_i \{d_i\}_i$) in k (we make the assumptions of the very beginning of §3.5), we set

$$\widehat{F}_{\varepsilon,\varphi}^P[G] := \widehat{F}_{q,\varphi}^P[G] \Big/ (q-\varepsilon)\widehat{F}_{q,\varphi}^P[G] \cong \widehat{F}_{q,\varphi}^P[G] \otimes_{k[q,q^{-1}]} k$$

(where k is meant as a $k[q,q^{-1}]$-algebra via $k \cong k[q,q^{-1}] \Big/ (q-\varepsilon)$); finally, from [CV-2], §3.3 (and also [DL], Proposition 6.4, for the one-parameter case) we record the existence of another *quantum Frobenius morphism*, namely a special Hopf algebra monomorphism

$$\widehat{\mathcal{F}r}_{G^\tau}\colon F[G^\tau] \cong \widehat{F}_{1,\varphi}^P[G] \hookrightarrow \widehat{F}_{\varepsilon,\varphi}^P[G] \tag{4.7}$$

(dual of $\widehat{\mathcal{F}r}_{\mathfrak{g}^\tau} \colon \widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{g}) \twoheadrightarrow \widehat{U}^Q_{1,\varphi}(\mathfrak{g}) \cong U(\mathfrak{g}^\tau)\,)$.

4.5 Completions of $F^M_{q,\varphi}[G]$. For later use, we also introduce some completions of quantum function algebras $F^M_{q,\varphi}[G]$. Let $\mathfrak{E}_\varphi := Ker\big(\varepsilon \colon F^M_{q,\varphi}[G] \rightarrow k(q)\big)$: we define $F^{M,\infty}_{q,\varphi}[G]$ to be the $\mathfrak{E}_\varphi$-adic completion of $F^M_{q,\varphi}[G]$, with its induced structure of topological Hopf algebra. The same construction work for integer forms: we define $\widehat{F}^{P,\infty}_{q,\varphi}[G]$, resp. $\widetilde{F}^{Q,\infty}_{q,\varphi}[G]$, to be the $\mathfrak{E}_\varphi$-adic completion of $\widehat{F}^P_{q,\varphi}[G]$, resp. $\widetilde{F}^Q_{q,\varphi}[G]$, where $\mathfrak{E}_\varphi := Ker\Big(\varepsilon \colon \widehat{F}^P_{q,\varphi}[G] \rightarrow k[q,q^{-1}]\Big)$, resp. $\mathfrak{E}_\varphi := Ker\Big(\varepsilon \colon \widetilde{F}^Q_{q,\varphi}[G] \rightarrow k[q,q^{-1}]\Big)$; both of these are topological Hopf algebras over $k[q,q^{-1}]$; finally, we define $\widetilde{F}^Q_{q,\varphi}[G]_{(q-1),\infty}$ to be the $(q-1)$-adic completion of $\widetilde{F}^Q_{q,\varphi}[G]$ (which is naturally embedded into $\widetilde{F}^{Q,\infty}_{q,\varphi}[G]$), which also is a topological Hopf algebra on its own. We shall prove later that all these are integer forms — in topological sense — of the corresponding Hopf algebras: for this reason we call all these new objects — the $F^{M,\infty}_{q,\varphi}[G]$ and their integer forms — **(multiparameter) quantum formal groups**.

§ 5 Quantum formal groups

5.1 Formal Hopf algebras and quantum formal groups. In this subsection we introduce the notion of *quantum formal group*. Recall (cf. [Di], ch. I) that formal groups can be defined in a category of a special type of commutative topological algebras, whose underlying vector space (or module) is linearly compact; following Drinfeld's philosophy,we define quantum formal groups by simply dropping out any commutativity assumption of the classical notion of formal group; thus we quickly outline how to modify the latter in order to define our new quantum objects, following the axiomatic construction of [Di], ch. I.

To begin with, let E be any vector space over a field K (one can then generalize more or less wathever follows to the case of free modules over a ring), and let E^* be its (linear) dual; we write $\langle x^*, x\rangle$ for $x^*(x)$ when $x \in E$, $x^* \in E^*$. We define on E^* the topology $\sigma(E^*, E)$ as the coarsest one such that for each $x \in E$ the linear map $x^* \mapsto \langle x^*, x\rangle$ of E^* into K is continuous, when K is given the discrete topology. A fundamental system of neighborhoods of 0 in E^* therefore consists of finite intersections of hyperplanes defined by equations $\langle x^*, x_j\rangle = 0$ for any finite sequence $\{x_j\}_j$ in E; we can describe this topology by choosing a basis $\{e_i\}_{i\in I}$ of E: to each $i \in I$ we associate the linear (coordinate) form e^*_i on E such that $\langle e^*_i, e_j\rangle = \delta_{ij}$, and we say that the family $\{e^*_i\}_{i\in I}$ is the *pseudobasis* of E^* dual to $\{e_i\}_{i\in I}$; then the subspace E' of E which is (algebraically) generated by the e^*_i is dense in E^*, and E^* is nothing but the *completion* of E', when E' is given the topology for which a fundamental system of neighborhoods of 0 consists of the vector subspaces containing almost all the e^*_i; thus elements of E^* can be described by series in the e^*_i's which in the given topology are in fact convergent. Finally, the topological vector spaces E^* are characterized by the property of linear compactness.

When dualizing twice, any linear form on E^* which is continuous for the topology $\sigma(E^*, E)$ is necessarily of type $x^* \mapsto \langle x^*, x\rangle$ for a unique $x \in E$, hence one gets back E as topological dual of E^*.

If now E, F are any two vector spaces over K, let E^*, F^* be their dual spaces, with

the topologies $\sigma(E^*, E)$, $\sigma(F^*, F)$; for any linear map $u: E \to F$, its transposed map $u^*: F^* \to E^*$ is continuous, and conversely for any linear map $v: F^* \to E^*$ which is continuous there exists a unique linear map $u: E \to F$ such that $v = u^*$.

When considering tensor products, $E^* \otimes F^*$ is naturally identified to a subspace of $(E \otimes F)^*$ by the formula $\langle x^* \otimes y^*, x \otimes y \rangle = \langle x^*, x \rangle \cdot \langle y^*, y \rangle$; this also shows that if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are bases of E and F, and $\{e_i^*\}_{i \in I}$ and $\{f_j^*\}_{j \in J}$ their dual pseudobases in E^* and F^*, then $\{e_i^* \otimes f_j^*\}_{i \in I, j \in J}$ is the dual pseudobasis of $\{e_i \otimes f_j\}_{i \in I, j \in J}$ in $(E \otimes F)^*$. This proves that $(E \otimes F)^*$ is the completion of $E^* \otimes F^*$ for the tensor product topology, i. e. the topology of $E^* \otimes F^*$ for which a fundamental system of neighborhoods of 0 consists of the sets $E^* \otimes V + W \otimes F^*$ where V, resp. W, ranges in a fundamental system of neighborhoods of 0 consisting in vector subspaces of F^*, resp. E^*; we denote this completion by $E^* \widehat{\otimes} F^*$, and we call it the *completed* (or *topological*) *tensor product* of E^* and F^*; the natural embedding of $E^* \otimes F^*$ into $(E \otimes F)^* = E^* \widehat{\otimes} F^*$ is then obviously continuous. Finally, when $u: E_1 \to E_2$, $v: F_1 \to F_2$ are linear maps, the transposed map $(u \otimes v)^*: (E_2 \otimes F_2)^* = {E_2}^* \widehat{\otimes} {F_2}^* \longrightarrow (E_1 \otimes F_1)^* = {E_1}^* \widehat{\otimes} {F_1}^*$ coincides with the continuous extension to ${E_2}^* \widehat{\otimes} {F_2}^*$ of the continuous map $u^* \otimes v^*: {E_2}^* \otimes {F_2}^* \to {E_1}^* \otimes {F_1}^*$; thus it is also denoted by $u^* \widehat{\otimes} v^*$.

We define a **linearly compact algebra** to be a topological algebra whose underlying vector space (or free module) is linearly compact (this is exactly the same definition of [Di], ch. I, §2.7, but for the fact that we do not require commutativity): then linearly compact algebras form a full subcategory of the category of topological algebras; morever, for any two objects A_1 and A_2 in this category, their topological tensor product $A_1 \widehat{\otimes} A_2$ is defined.

Dually, within the category of linearly compact vector spaces we define **linearly compact coalgebras** as triplets (C, Δ, ε) with $\Delta: C \to C \widehat{\otimes} C$ and $\varepsilon: C \to K$ satisfying the usual coalgebra axioms. Then one checks (with the same arguments as [Di], which never need neither commutativity nor cocommutativity) that $(\ )^*: (A, m, 1) \mapsto (A^*, m^*, 1^*)$ defines a contravariant functor from algebras to linearly compact coalgebras, while
$(\ )^*: (C, \Delta, \varepsilon) \mapsto (C^*, \Delta^*, \varepsilon^*)$ defines a contravariant functor from coalgebras to linearly compact algebras.

Finally, we define a **formal Hopf algebra** as a datum $(H, m, 1, \Delta, \varepsilon, S)$ such that $(H, m, 1)$ is a linearly compact algebra, (H, Δ, ε) is a linearly compact coalgebra, and the usual compatibility axioms of Hopf algebras are satisfied. In the following, we shall also consider "usual" Hopf algebras as particular formal Hopf algebras.

According to Drinfeld's dictionary, we define **quantum formal group** the *spectrum* of a formal Hopf algebra (whereas *classical* formal groups are spectra of *commutative* formal Hopf algebra: cf. [Di], ch. I).

We will now use the language of quantum formal groups to construct a suitable dual of Drinfeld's double:

Proposition 5.2. *Let H_-, H_+ be Hopf F-algebras, let $\pi: (H_-)_{op} \otimes H_+ \to F$ be an arbitrary Hopf pairing, and let $D := D(H_-, H_+, \pi)$ be the corresponding Drinfeld's double. Then there exist F-algebra isomorphisms*

$$D^* \cong {H_+}^* \widehat{\otimes} {H_-}^* , \quad D^* \cong {H_-}^* \widehat{\otimes} {H_+}^*$$

dual of the F-coalgebra isomorphisms $D \cong H_+ \otimes H_-$, $D \cong H_- \otimes H_+$ provided by multiplication (cf. §3.1).

Proof. It follows from the previous discussion that dualizing (3.1) yields vector space isomorphisms

$$D^* \cong {H_+}^* \,\widehat{\otimes}\, {H_-}^* \,, \quad D^* \cong {H_-}^* \,\widehat{\otimes}\, {H_+}^* \,;$$

then a straightforward computation shows that this is an isomorphism of algebras, the right-hand-side being thought of as a topological tensor product *of algebras*. $\square$

We apply all this to quantum groups. To begin with, consider the Drinfeld's double $D_{q,\varphi}^M(\mathfrak{g})$, identified as a coalgebra with $U_{q,\varphi}^M(\mathfrak{b}_+) \otimes U_{q,\varphi}^Q(\mathfrak{b}_-)$; this is a Hopf algebra, whose dual is the formal Hopf algebra ${D_{q,\varphi}^Q(\mathfrak{g})}^*$ isomorphic to ${U_{q,\varphi}^M(\mathfrak{b}_+)}^* \,\widehat{\otimes}\, {U_{q,\varphi}^Q(\mathfrak{b}_-)}^*$.

From DRT pairings one gets natural Hopf algebra embeddings $U_{q,\varphi}^{\overset{P}{Q}}(\mathfrak{b}_\mp) \hookrightarrow {U_{q,\varphi}^{\overset{Q}{P}}(\mathfrak{b}_\pm)}^*$; from these we get algebra embeddings $U_{q,\varphi}^{\overset{P}{Q}}(\mathfrak{b}_-) \,\widehat{\otimes}\, U_{q,\varphi}^P(\mathfrak{b}_+) \hookrightarrow {U_{q,\varphi}^{\overset{Q}{P}}(\mathfrak{b}_+)}^* \,\widehat{\otimes}\, {U_{q,\varphi}^Q(\mathfrak{b}_-)}^*$, hence we will identify $U_{q,\varphi}^{\overset{P}{Q}}(\mathfrak{b}_-) \,\widehat{\otimes}\, U_{q,\varphi}^P(\mathfrak{b}_+)$ with its image in ${U_{q,\varphi}^{\overset{Q}{P}}(\mathfrak{b}_+)}^* \,\widehat{\otimes}\, {U_{q,\varphi}^Q(\mathfrak{b}_-)}^*$.

Lemma 5.3. *The algebra* $U_{q,\varphi}^{\overset{P}{Q}}(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q^P(\mathfrak{b}_+)$ *is a formal Hopf subalgebra of* ${D_{q,\varphi}^{\overset{Q}{P}}(\mathfrak{g})}^* = {U_{q,\varphi}^{\overset{Q}{P}}(\mathfrak{b}_+)}^* \,\widehat{\otimes}\, {U_{q,\varphi}^Q(\mathfrak{b}_-)}^*$.

Proof. For the sake of simplicity we stick to $\varphi = 0$, the general case being entirely similar.

It is known (cf. [Lu-1], [APW], [Tk]) that the dual Hopf algebra of the quantized enveloping algebra $U_{q,0}^Q(\mathfrak{g}) = U_q^Q(\mathfrak{g})$ is

$${U_q^Q(\mathfrak{g})}^\circ \cong F_q^P[G] \rtimes k(q)[{\mathbb{Z}_2}^n]$$

where the symbol $\rtimes$ denotes a semidirect product of Hopf algebras and $k(q)[{\mathbb{Z}_2}^n]$ is the (Hopf) group algebra of the multiplicative group ${\mathbb{Z}_2}^n$ (with $\mathbb{Z}_2 = \{-1,+1\}$), acting on $U_q^Q(\mathfrak{g})$ by $(\nu_i)_i(E_j) := 0$, $(\nu_i)_i(K_j) := \nu_j$, $(\nu_i)_i(F_j) := 0$ for all $(\nu_i)_i \in {\mathbb{Z}_2}^n$. Similarly, letting $k^\star := k \setminus \{0\}$, one has

$$\begin{aligned} {U_q^Q(\mathfrak{t})}^\circ &\cong U_q^P(\mathfrak{t}) \rtimes k(q)[(k^\star)^n] \\ {U_q^Q(\mathfrak{b}_-)}^\circ &\cong U_q^P(\mathfrak{b}_+) \rtimes k(q)[(k^\star)^n] \\ {U_q^Q(\mathfrak{b}_+)}^\circ &\cong U_q^P(\mathfrak{b}_-) \rtimes k(q)[(k^\star)^n] \end{aligned}$$

(the multiplicative group $(k^\star)^n$, acting on $U_q^Q(\mathfrak{g})$ by $(\nu_i)_i(E_j) := 0$, $(\nu_i)_i(K_j) := \nu_j$, $(\nu_i)_i(F_j) := 0$ for all $(\nu_i)_i \in (k^\star)^n$), which in short hold because

$$U_q^Q(\mathfrak{t})^\circ = k(q)[Q]^\circ \cong k(q)[X(Q)] = k(q)[P \rtimes (k^\star)^n] = U_q^P(\mathfrak{t}) \rtimes k(q)[(k^\star)^n]$$

(where $X(Q)$ is the group of characters $Q \to k(q)^\star$, clearly $X(Q) \cong (k(q)^\star)^n \cong P \rtimes (k^\star)^n$ with the action of $\nu = (\nu_1, \dots, \nu_n) \in (k^\star)^n$ given by $\langle \nu, K_i \rangle = \nu_i$, and we use the well-known fact $K[\Gamma]^\circ \cong K[X(\Gamma)]$ for all fields K and all commutative groups Γ), and quantum Borel algebras admit triangular decomposition. It is immediate to check — looking at PBW bases and using [Sw], Theorem 6.1.3, namely: If H is a commutative Hopf algebra which is finitely generated (as an algebra), then H° is dense in H^* — that ${U_q^Q(\mathfrak{b}_\pm)}^\circ \cong U_q^P(\mathfrak{b}_\mp) \rtimes k(q)[(k^\star)^n]$ is dense in ${U_q^Q(\mathfrak{b}_\pm)}^*$, so that

$$U_q^Q(\mathfrak{b}_+)^* \cong U_q^P(\mathfrak{b}_-) \widehat{\rtimes} k(q)[(k^\star)^n]\,, \quad U_q^Q(\mathfrak{b}_-)^* \cong U_q^P(\mathfrak{b}_+) \widehat{\rtimes} k(q)[(k^\star)^n]$$

(the symbols $\widehat{\rtimes}$ denoting the obvious formal series completion); it follows that

$$D_q^{\overset{Q}{P}}(\mathfrak{g})^* \cong \left(U_q^P(\mathfrak{b}_-) \widehat{\rtimes} k(q)[(k^\star)^n] \right) \widehat{\otimes} \left(U_q^P(\mathfrak{b}_+) \widehat{\rtimes} k(q)[(k^\star)^n] \right) .$$

Then looking at definitions one immediately convinces himself that

$$\Delta \left(U_q^{\overset{P}{Q}}(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q^P(\mathfrak{b}_+) \right) \subseteq \left(U_q^{\overset{P}{Q}}(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q^P(\mathfrak{b}_+) \right) \widehat{\otimes} \left(U_q^{\overset{P}{Q}}(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q^P(\mathfrak{b}_+) \right)$$

$$S \left(U_q^{\overset{P}{Q}}(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q^P(\mathfrak{b}_+) \right) = U_q^{\overset{P}{Q}}(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q^P(\mathfrak{b}_+)$$

whence the thesis. $\square$

In light of the previous lemma, the non-degenerate canonical pairing $D_{q,\varphi}^{\overset{Q}{P}}(\mathfrak{g})^* \otimes D_{q,\varphi}^{\overset{Q}{P}}(\mathfrak{g}) \to k(q)$ given by evaluation restricts to a non-degenerate pairing

$$\left(U_{q,\varphi}^{\overset{P}{Q}}(\mathfrak{b}_-) \,\widehat{\otimes}\, U_{q,\varphi}^P(\mathfrak{b}_+) \right) \otimes D_{q,\varphi}^{\overset{Q}{P}}(\mathfrak{g}) = \left(U_{q,\varphi}^{\overset{P}{Q}}(\mathfrak{b}_-) \,\widehat{\otimes}\, U_{q,\varphi}^P(\mathfrak{b}_+) \right) \otimes \left(U_{q,\varphi}^{\overset{Q}{P}}(\mathfrak{b}_+) \otimes U_{q,\varphi}^Q(\mathfrak{b}_-) \right) \to k(q)$$

of formal Hopf algebras, which is defined by

$$\langle f \otimes g, x \otimes y \rangle = \langle f, x \rangle_{\pi_\mp^\varphi} \cdot \langle g, y \rangle_{\overline{\pi_-^\varphi}} = \pi_\mp^\varphi(f,x) \cdot \overline{\pi_-^\varphi}(g,y)$$

for all $f \in U_{q,\varphi}^{\overset{P}{Q}}(\mathfrak{b}_-)$, $g \in U_{q,\varphi}^P(\mathfrak{b}_+)$, $x \in U_{q,\varphi}^{\overset{Q}{P}}(\mathfrak{b}_+)$, $y \in U_{q,\varphi}^Q(\mathfrak{b}_-)$.

Now consider root vectors F_α, E_α ($\alpha \in R^+$) defined in §2.4: we set

$$F_\alpha^\varphi := K_{\tau_\alpha} F_\alpha = F_\alpha K_{\tau_\alpha}\,, \quad E_\alpha^\varphi := K_{\tau_\alpha} E_\alpha = E_\alpha K_{\tau_\alpha}$$

$$\overline{F}_\alpha^\varphi := K_{\tau_\alpha} \overline{F}_\alpha = (q_\alpha - q_\alpha^{-1}) F_\alpha^\varphi\,, \quad \overline{E}_\alpha^\varphi := K_{\tau_\alpha} \overline{E}_\alpha = (q_\alpha - q_\alpha^{-1}) E_\alpha^\varphi$$

and $E_i^\varphi := E_{\alpha_i}^\varphi$, $F_i^\varphi := F_{\alpha_i}^\varphi$, $\overline{E}_i^\varphi := \overline{E}_{\alpha_i}^\varphi$, $\overline{F}_i^\varphi := \overline{F}_{\alpha_i}^\varphi$ for all i (cf. [CV-2], §1.7). From now on, we let M be any lattice such that $Q \leq M \leq P$, and M' its dual lattice.

Definition 5.4. *We call $A_\infty^{\varphi,M}$ the closed topological subalgebra of $U_{q,\varphi}^M(\mathfrak{b}_-) \,\widehat{\otimes}\, U_{q,\varphi}^P(\mathfrak{b}_+)$ generated by*

$$\{\, F_\alpha^\varphi \otimes 1, L_{-(1+\varphi)(\lambda)} \otimes L_{(1-\varphi)(\lambda)}, 1 \otimes E_\alpha^\varphi \mid \alpha \in R^+; \lambda \in M \,\}$$

that is, $A_\infty^{\varphi,M}$ is the closure in $U_{q,\varphi}^M(\mathfrak{b}_-) \,\widehat{\otimes}\, U_{q,\varphi}^P(\mathfrak{b}_+)$ of the subalgebra $A^{\varphi,M}$ generated by

$$\{\, F_\alpha^\varphi \otimes 1, L_{-(1+\varphi)(\lambda)} \otimes L_{(1-\varphi)(\lambda)}, 1 \otimes E_\alpha^\varphi \mid \alpha \in R^+; \lambda \in M \,\}\,. \ \square$$

Proposition 5.5. *The algebra $A_\infty^{\varphi,M}$ is the subset $\mathcal{A}_\infty^{\varphi,M}$ in $U_{q,\varphi}^M(\mathfrak{b}_-) \,\widehat{\otimes}\, U_{q,\varphi}^P(\mathfrak{b}_+)$ $\left(\subset D_{q,\varphi}^{M'}(\mathfrak{g})^* \right)$ of functions on $D_{q,\varphi}^{M'}(\mathfrak{g})$ which vanish on $\mathfrak{K}_\varphi^{M'}$. In particular, $A_\infty^{\varphi,M}$ is a formal Hopf subalgebra of $U_{q,\varphi}^{M'}(\mathfrak{b}_+)^* \,\widehat{\otimes}\, U_{q,\varphi}^Q(\mathfrak{b}_-)^*$.*

Proof. (cf. [DL], §4.2, and [CV-2], Lemma 2.5) Again we give for simplicity the proof for $\varphi = 0$, the general case arising exactly in the same way putting φ throughout.

From the very definition of $\mathfrak{K}^{M'} := \mathfrak{K}_0^{M'}$ (cf. §3.3) it is clear that $\mathcal{A}_\infty^M := \mathcal{A}_\infty^{0,M}$ is nothing but the subset of $\phi \in U_q^M(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q^P(\mathfrak{b}_+)$ enjoying the property

$$\phi(EL \otimes NF) = \phi(ELN \otimes F) = \phi(E \otimes LNF) \ , \tag{5.1}$$

for all $E \in U_q(\mathfrak{n}_+)$, $L, N \in U_q^{M'}(\mathfrak{t})$, $F \in U_q(\mathfrak{n}_-)$; thus we have to show that A_∞^M is exactly the subset of functionals in $U_q^P(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q^P(\mathfrak{b}_+)$ enjoying property (5.1).

First of all A_∞^M have this property, for any element of A_∞^M is a linear combination of elements $FK_{-\lambda} \otimes K_\lambda E$ where F, E are monomials in the F_i's, E_i's; thus we have

$$\langle FK_{-\lambda} \otimes K_\lambda E, E'L \otimes NF' \rangle = \pi_-(F,E') \cdot \pi_-(K_{-\lambda}, L) \cdot \overline{\pi_-}(K_\lambda, N) \cdot \overline{\pi_-}(E,F') \tag{5.2}$$

for all $E' \in U_q(\mathfrak{n}_+)$, $L, N \in U_q^{M'}(\mathfrak{t})$, $F' \in U_q(\mathfrak{n}_-)$. If we look at K_λ, $\lambda \in M$, as characters of the algebra $U_q^{M'}(\mathfrak{t})$, letting $L(\lambda) = \pi_-(K_{-\lambda}, L) \ \forall \, L \in U_q^{M'}(\mathfrak{t})$, then (5.2) reads

$$\langle FK_{-\lambda} \otimes K_\lambda E, E'L \otimes NF' \rangle = \pi_-(F,E') \, \overline{\pi_-}(E,F') \, L(\lambda) \, N(\lambda) \ ; \tag{5.3}$$

similarly we find

$$\langle FK_{-\lambda} \otimes K_\lambda E, E' \otimes LNF' \rangle = \pi_-(F,E') \, \overline{\pi_-}(E,F') \, 1(\lambda) \, LN(\lambda) \tag{5.4}$$

$$\langle FK_{-\lambda} \otimes K_\lambda E, E'LN \otimes F' \rangle = \pi_-(F,E') \, \overline{\pi_-}(E,F') \, LN(\lambda) \, 1(\lambda) \tag{5.5}$$

and right-hand-side of (5.3–4–5) are clearly equal.

On the other hand, any $\phi \in U_q^M(\mathfrak{b}_-) \otimes U_q^P(\mathfrak{b}_+)$ is a (possibly infinite) linear combination of elements $FK_{-\sigma} \otimes K_\lambda E$; as above computation gives

$$\langle FK_{-\sigma} \otimes K_\lambda E, E'L \otimes NF' \rangle = \pi_-(F,E') \, \overline{\pi_-}(E,F') \, L(\sigma) \, N(\lambda)$$
$$\langle FK_{-\sigma} \otimes K_\lambda E, E' \otimes LNF' \rangle = \pi_-(F,E') \, \overline{\pi_-}(E,F') \, LN(\lambda)$$
$$\langle FK_{-\sigma} \otimes K_\lambda E, E'LN \otimes F' \rangle = \pi_-(F,E') \, \overline{\pi_-}(E,F') \, LN(\sigma)$$

thus (5.1) holds if and only if $\sigma = \lambda$.

Now consider the Hopf structure. Since $\mathfrak{K}^{M'}$ is a Hopf ideal we have $S(\mathfrak{K}^{M'}) = \mathfrak{K}^{M'}$, whence $\langle S(f), \mathfrak{K}^{M'} \rangle = \langle f, S(\mathfrak{K}^{M'}) \rangle = 0$ for all $f \in A_\infty^M$, so that $S(A_\infty^M) = A_\infty^M$. On the other hand, let $f \in A_\infty^M$ with $\Delta(f) = \sum_n g_n \otimes h_n \in \big(U_q^M(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q^P(\mathfrak{b}_+)\big) \,\widehat{\otimes}\, \widehat{\otimes} \, \big(U_q^M(\mathfrak{b}_-) \,\widehat{\otimes}\, U_q^P(\mathfrak{b}_+)\big) = \big(U_q^M(\mathfrak{b}_-) \otimes U_q^P(\mathfrak{b}_+)\big) \,\widehat{\otimes}\, \big(U_q^M(\mathfrak{b}_-) \otimes U_q^P(\mathfrak{b}_+)\big)$ (with n ranging in a possibly infinite set of indices); we can take a similar expression so that $g_n \neq g_{n'}$ for $n \neq n'$, and for each fixed factor $g_{\overline{n}}$ there exists an $x_{\overline{n}} \in U_q^Q(\mathfrak{b}_+) \otimes U_q^Q(\mathfrak{b}_-)$ such that $\langle g_n, x_{\overline{n}} \rangle \neq 0$ if and only if $n = \overline{n}$; then for any fixed $g_{\overline{n}}$ we have

$$0 = \langle f, x_{\overline{n}} \cdot y \rangle = \langle \Delta(f), x_{\overline{n}} \otimes y \rangle = \textstyle\sum_n \langle g_n, x_{\overline{n}} \rangle \cdot \langle h_n, y \rangle = \langle g_{\overline{n}}, x_{\overline{n}} \rangle \cdot \langle h_{\overline{n}}, y \rangle$$

for all $y \in \mathfrak{K}^{M'}$, whence $h_{\overline{n}} \in \mathcal{A}_\infty^M = A_\infty^M$; we conclude that $h_n \in A_\infty^M$ for all n, and similarly for g_n, hence $\Delta(A_\infty^M) \leq A_\infty^M \otimes A_\infty^M$. The thesis follows. $\square$

5.6 *Remark:* notice that $A_\infty^{\varphi,M}$ can also be defined — as an *algebra* — as the $\mathfrak{E}'$-adic completion of the subalgebra $A^{\varphi,M}$, where $\mathfrak{E}' := \mathfrak{E} \cap A^{\varphi,M}$ with $\mathfrak{E} := Ker\big(\varepsilon \colon A_\infty^{\varphi,M} \to k(q)\big)$; thus $A_\infty^{\varphi,M}$ is a formal Hopf algebra, complete in the $\mathfrak{E}$-adic topology.

We now present one of the crucial steps, namely the realization of the quantum function algebra $F_{q,\varphi}^M[G]$ as a formal Hopf subalgebra of $A_\infty^{\varphi,M}$, thus improving [CV-2], Lemma 2.5 (which extends [DL], Lemma 4.3, to the multiparameter case).

Theorem 5.7. *Let $pr_{\varphi}^{M'} \colon D_{q,\varphi}^{M'}(\mathfrak{g}) \twoheadrightarrow U_{q,\varphi}^{M'}(\mathfrak{g})$ be the canonical epimorphism of §3.3. Then dualization gives an embedding of formal Hopf algebras*

$$\left(pr_{\varphi}^{M'}\right)^{*} \colon U_{q,\varphi}^{M'}(\mathfrak{g})^{*} \hookrightarrow D_{q,\varphi}^{M'}(\mathfrak{g})^{*} \cong U_{q,\varphi}^{M'}(\mathfrak{b}_{+})^{*} \,\widehat{\otimes}\, U_{q,\varphi}^{M'}(\mathfrak{b}_{-})^{*} \ ;$$

the restriction of $\left(pr_{\varphi}^{M'}\right)^{*}$ gives two embeddings of formal Hopf algebras, namely $\mu_{\varphi}^{M} \colon F_{q,\varphi}^{M}[G] \hookrightarrow A_{\infty}^{\varphi,M}$, whose image is contained in $A^{\varphi,M}$, and $\mu_{\infty}^{\varphi,M} \colon F_{q,\varphi}^{M,\infty}[G] \hookrightarrow A_{\infty}^{\varphi,M}$, the second being continuous extension of the first.

Proof. The statements about restrictions are the unique non-trivial facts to prove. Recall (cf. §3.1) that the identification $D_{q,\varphi}^{M'}(\mathfrak{g}) = U_{q,\varphi}^{M'}(\mathfrak{b}_{+}) \otimes U_{q,\varphi}^{M'}(\mathfrak{b}_{-})$ is given by the composition

$$U_{q,\varphi}^{M'}(\mathfrak{b}_{+}) \otimes U_{q,\varphi}^{M'}(\mathfrak{b}_{-}) \overset{j_{+} \otimes j_{-}}{\hookrightarrow} D_{q,\varphi}^{M'}(\mathfrak{g}) \otimes D_{q,\varphi}^{M'}(\mathfrak{g}) \xrightarrow{m_D} D_{q,\varphi}^{M'}(\mathfrak{g})$$

where $j_{\pm} \colon U_{q,\varphi}^{M'}(\mathfrak{b}_{\pm}) \hookrightarrow D_{q,\varphi}^{M'}(\mathfrak{g})$ are the natural Hopf embeddings, $m_D \colon D_{q,\varphi}^{M'} \otimes D_{q,\varphi}^{M'}(\mathfrak{g}) \longrightarrow D_{q,\varphi}^{M'}(\mathfrak{g})$ is the multiplication of $D_{q,\varphi}^{M'}(\mathfrak{g})$ and we look at this composition as a Hopf algebra isomorphism (thus pulling back the Hopf structure on $U_{q,\varphi}^{M'}(\mathfrak{b}_{+}) \otimes U_{q,\varphi}^{M'}(\mathfrak{b}_{-})$); by duality, the identification $D_{q,\varphi}^{M'}(\mathfrak{g})^{*} = U_{q,\varphi}^{M'}(\mathfrak{b}_{+})^{*} \,\widehat{\otimes}\, U_{q,\varphi}^{M'}(\mathfrak{b}_{-})^{*}$ is given by $\left(m_D \circ (j_{+} \otimes j_{-})\right)^{*} = \left(j_{+}^{*} \,\widehat{\otimes}\, j_{-}^{*}\right) \circ m_D{}^{*}$. If m_U is the multiplication of $U_{q,\varphi}^{Q}(\mathfrak{g})$, we have $m_U \circ pr^{M'} \otimes pr^{M'} = pr^{M'} \circ m_D$, hence dualizing

$$U_{q,\varphi}^{M'}(\mathfrak{b}_{+}) \otimes U_{q,\varphi}^{M'}(\mathfrak{b}_{-}) \overset{j_{+} \otimes j_{-}}{\hookrightarrow} D_{q,\varphi}^{M'}(\mathfrak{g}) \otimes D_{q,\varphi}^{M'}(\mathfrak{g}) \overset{m_D}{\to} D_{q,\varphi}^{M'}(\mathfrak{g}) \overset{pr^{M'}}{\twoheadrightarrow} U_{q,\varphi}^{M'}(\mathfrak{g})$$

yields $\left(pr^{M'} \circ m_D \circ (j_{+} \otimes j_{-})\right)^{*} = \left(pr^{M'} \circ j_{+}\right)^{*} \,\widehat{\otimes}\, \left(pr^{M'} \circ j_{-}\right)^{*} \circ m_U^{*}$; but $pr^{M'} \circ j_{\pm}$ coincides with the natural embedding $i_{\pm} \colon U_{q,\varphi}^{M'}(\mathfrak{b}_{\pm}) \hookrightarrow U_{q,\varphi}^{M'}(\mathfrak{g})$, thus $\left(pr^{M'} \circ m_U \circ (j_{+} \otimes j_{-})\right)^{*} = = \left(i_{+}^{*} \,\widehat{\otimes}\, i_{-}^{*}\right) \circ m_U^{*}$. Now m_U^{*} is nothing but the comultiplication of $U_{q,\varphi}^{M'}(\mathfrak{g})^{*}$, which restricts to the comultiplication Δ of $F_{q,\varphi}^{M}[G]$, while $\rho_{\pm} := i_{\pm}^{*} \colon U_{q,\varphi}^{M'}(\mathfrak{g})^{*} \longrightarrow U_{q,\varphi}^{M'}(\mathfrak{b}_{\pm})^{*}$ is the "restriction" map, which maps $F_{q,\varphi}^{M}[G]$ to $F_{q,\varphi}^{M}[B_{\pm}]$; therefore

$$\begin{aligned} \left(pr^{M'} \circ m_U \circ (j_{+} \otimes j_{-})\right)^{*} \left(F_{q,\varphi}^{M}[G]\right) &= \left(i_{+}^{*} \,\widehat{\otimes}\, i_{-}^{*}\right) \left(\Delta\left(F_{q,\varphi}^{P}[G]\right)\right) \subseteq \\ &\subseteq \rho_{+}(F_{q,\varphi}^{P}[G]) \otimes \rho_{-}(F_{q,\varphi}^{P}[G]) = F_{q,\varphi}^{P}[B_{+}] \otimes F_{q,\varphi}^{P}[B_{-}] \cong U_{q,\varphi}^{P}(\mathfrak{b}_{-}) \otimes U_{q,\varphi}^{P}(\mathfrak{b}_{+}) \ ; \end{aligned}$$

we conclude that the embedding $U_{q,\varphi}^{M'}(\mathfrak{g})^{*} \hookrightarrow D_{q,\varphi}^{M'}(\mathfrak{g})^{*} \cong U_{q,\varphi}^{M'}(\mathfrak{b}_{p})^{*} \,\widehat{\otimes}\, U_{q,\varphi}^{M'}(\mathfrak{b}_{-})^{*}$ maps $F_{q,\varphi}^{M}[G]$ into $U_{q,\varphi}^{M}(\mathfrak{b}_{-}) \otimes U_{q,\varphi}^{P}(\mathfrak{b}_{+})$. From the very definitions it follows that $\mu^{M}(F_{q,\varphi}^{M}[G])$ vanishes on $\mathfrak{K}^{P}$, hence (Proposition 5.5) we conclude that $\mu^{M}\left(F_{q,\varphi}^{M}[G]\right) \subseteq A^{\varphi,M}$.

Finally, from what we already proved it follows that the $\mathfrak{E}_F$ (the kernel of $\varepsilon \colon F_{q,\varphi}^{M}[G] \to k(q)$) is exactly the preimage of $\mathfrak{E}_A$ (the kernel of $\varepsilon \colon A_{\infty}^{\varphi,M} \to k(q)$), whence — because of §4.5 and §5.6 — the map $\mu_{\infty}^{M} \colon F_{q,\varphi}^{M,\infty}[G] \to A_{\infty}^{\varphi,M}$ coincides with the continuous extension of $\mu^{M} \colon F_{q,\varphi}^{M}[G] \to A_{\infty}^{\varphi,M}$, as claimed. □

5.8 Integer forms of $A_{\infty}^{\varphi,M}$. In order to specialize our quantum objects, we need integer forms. Since $A_{\infty}^{\varphi,M} \leq \left(pr^{M'}\right)^{*} \left(U_{q,\varphi}^{M'^{*}}\right)$ (Proposition 5.5), $A_{\infty}^{\varphi,M}$ can be thought of as a formal Hopf algebra of linear functions on $U_{q,\varphi}^{M'}$; therefore we define integer forms

of $A_\infty^{\varphi,M}$ as dual of integer forms of $U_{q,\varphi}^{M'}$, that is

$$\widehat{A}_\infty^{\varphi,P} := \left\{ f \in A_\infty^{\varphi,P} \,\middle|\, \left\langle f, \widehat{U}_{q,\varphi}^{Q}(\mathfrak{g}) \right\rangle \subseteq k[q,q^{-1}] \right\}$$

$$\widetilde{A}_\infty^{\varphi,Q} := \left\{ f \in A_\infty^{\varphi,Q} \,\middle|\, \left\langle f, \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) \right\rangle \subseteq k[q,q^{-1}] \right\}.$$

In the sequel, for a *pseudobasis* we will mean a complete system of linear generators which are linearly independent (i. e. a basis in the sense of topological free modules).

Proposition 5.9. *The algebra $\widehat{A}_\infty^{\varphi,P}$ is topologically generated by*

$$\left\{ \overline{F}^{\varphi}_{\alpha^h} \otimes 1, L_{-(1+\varphi)(\lambda)} \otimes L_{(1-\varphi)(\lambda)}, 1 \otimes \overline{E}^{\varphi}_{\alpha^k} \,\middle|\, h,k = 1,\dots,N; \lambda \in P_+ \right\}$$

and has the $k[q,q^{-1}]$-pseudobasis of PBW type

$$\left\{ \prod_{h=N}^{1} \left(\overline{F}^{\varphi}_{\alpha^h}\right)^{a_h} \cdot L_{-(1+\varphi)(\lambda)} \otimes L_{(1-\varphi)(\lambda)} \cdot \prod_{k=1}^{N} \left(\overline{E}^{\varphi}_{\alpha^k}\right)^{d_k} \,\middle|\, a_h, d_k \in \mathbb{N}; \lambda \in P_+ \right\}; \qquad (5.6)$$

in particular, it is a $k[q,q^{-1}]$-form of the formal Hopf algebra $A_\infty^{\varphi,P}$.

(b) The algebra $\widetilde{A}_\infty^{\varphi,Q}$ is topologically generated by

$$\left\{ \left(F^{\varphi}_{\alpha^h}\right)^{(a)} \otimes 1, \begin{pmatrix} K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}; c \\ t \end{pmatrix}, 1 \otimes \left(E^{\varphi}_{\alpha^k}\right)^{(d)} \,\middle|\, \forall\, h,k,i,\, a,t,d \in \mathbb{N},\, c \in \mathbb{Z} \right\}$$

and has the $k[q,q^{-1}]$-pseudobasis of PBW type

$$\left\{ \prod_{h=N}^{1} \left(F^{\varphi}_{\alpha^h}\right)^{(a_h)} \cdot \prod_{i=1}^{n} \begin{pmatrix} K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}; 0 \\ t_i \end{pmatrix} \cdot \prod_{k=1}^{N} \left(E^{\varphi}_{\alpha^k}\right)^{(d_k)} \,\middle|\, a_h, t_i, d_k \in \mathbb{N} \right\}; \qquad (5.7)$$

in particular, it is a $k[q,q^{-1}]$-form of the formal Hopf algebra $A_\infty^{\varphi,Q}$. $\square$

Proof. (a) It is clear that (5.6) is a (PBW- type) $k(q)$-basis of $A_\infty^{\varphi,P}$; thus, given $f \in A_\infty^{\varphi,P}$, write it as a (possibly infinite) sum

$$f = \sum_\sigma a_\sigma \cdot \overline{F}^{\varphi}_\sigma \cdot L^{\varphi,-}_\sigma \otimes L^{\varphi,+}_\sigma \cdot \overline{E}^{\varphi}_\sigma$$

where $a_\sigma \in k(q)$ and the $\overline{F}^{\varphi}_\sigma$'s, resp. the $L^{\varphi,-}_\sigma \otimes L^{\varphi,+}_\sigma$'s, resp. the $\overline{E}^{\varphi}_\sigma$'s, are ordered monomials in the $\overline{F}^{\varphi}_{\alpha^k}$'s, resp. in the $L_{-(1+\varphi)(\lambda)} \otimes L_{(1-\varphi)(\lambda)}$'s, resp. in the $\overline{E}^{\varphi}_{\alpha^k}$'s ($k = 1,\dots,N$, $\lambda \in P$). For any monomial of divided powers $\widehat{E} \cdot (K) \cdot \widehat{F}$ in the PBW basis of $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{g})$ we have

$$\left\langle f, \widehat{E} \cdot (K) \cdot \widehat{F} \right\rangle = \sum_\nu a_\sigma \cdot \left\langle \overline{F}^{\varphi}_\sigma \cdot L^{\varphi,-}_\sigma \otimes L^{\varphi,+}_\sigma \cdot \overline{E}^{\varphi}_\sigma, \widehat{E} \cdot (K) \otimes 1 \cdot \widehat{F} \right\rangle =$$

$$= \sum_\sigma a_\sigma \cdot \left\langle \overline{F}^{\varphi}_\sigma, \widehat{E} \right\rangle_{\pi^\varphi_-} \cdot \left\langle L^{\varphi,-}_\sigma, (K) \right\rangle_{\pi^\varphi_-} \cdot \left\langle L^{\varphi,+}_\sigma, 1 \right\rangle_{\overline{\pi^\varphi_-}} \cdot \left\langle \overline{E}^{\varphi}_\sigma, \widehat{F} \right\rangle_{\overline{\pi^\varphi_-}}$$

For any fixed $\bar{\sigma}$, there exist divided powers monomials $\widehat{E}_{\bar{\sigma}}$, $\widehat{F}_{\bar{\sigma}}$ such that $\left\langle \overline{F}^{\varphi}_{\bar{\sigma}}, \widehat{E}_{\bar{\sigma}} \right\rangle_{\pi^{\varphi}_{-}} = = \pm q^r$, $\left\langle \overline{E}^{\varphi}_{\bar{\sigma}}, \widehat{F}_{\bar{\sigma}} \right\rangle_{\overline{\pi^{\varphi}_{-}}} = \pm q^s$ for some $r, s \in \mathbb{Z}$, and $\left\langle \overline{F}^{\varphi}, \widehat{E}_{\bar{\sigma}} \right\rangle_{\pi^{\varphi}_{-}} = 0$, $\left\langle \overline{E}^{\varphi}, \widehat{F}_{\bar{\sigma}} \right\rangle_{\overline{\pi^{\varphi}_{-}}} = 0$ for *all* PBW monomials $\overline{F}^{\varphi} \neq \overline{F}^{\varphi}_{\bar{\sigma}}$ and $\overline{E}^{\varphi} \neq \overline{E}^{\varphi}_{\bar{\sigma}}$ (cf. (2.2)); then

$$\left\langle f, \widehat{E}_{\bar{\sigma}} \cdot (K) \cdot \widehat{F}_{\bar{\sigma}} \right\rangle = \textstyle\sum_{\sigma \in \mathcal{S}} a_\sigma \epsilon_\sigma q^{r+s} \cdot \left\langle L^{\varphi,-}_{\sigma}, (K) \right\rangle_{\pi^{\varphi}_{-}}$$

(where $\mathcal{S} := \left\{ \sigma \,\middle|\, \overline{F}^{\varphi}_{\sigma} = \overline{F}^{\varphi}_{\bar{\sigma}}, \overline{E}^{\varphi}_{\sigma} = \overline{E}^{\varphi}_{\bar{\sigma}} \right\}$ and $\epsilon_\sigma = \pm 1$) which can be rewritten as

$$\left\langle \widehat{F}_{\bar{\sigma}} \triangleright f \triangleleft \widehat{E}_{\bar{\sigma}}, (K) \otimes 1 \right\rangle = \textstyle\sum_{\sigma \in \mathcal{S}} a_\sigma \epsilon_\sigma q^{r+s} \cdot \left\langle L^{\varphi,-}_{\sigma}, (K) \right\rangle_{\pi_{-}}$$

(where $\triangleright$, resp. $\triangleleft$, denotes the left, resp. right, action of $U^Q_{q,\varphi}(\mathfrak{g})$ onto $U^Q_{q,\varphi}(\mathfrak{g})^*$); therefore

$$\left(\widehat{F}_{\bar{\sigma}} \triangleright f \triangleleft \widehat{E}_{\bar{\sigma}} \right) \Big|_{\widehat{U}^Q_{q,\varphi}(\mathfrak{t})^{\otimes 2}} = \sum_{\sigma \in \mathcal{S}} a_\sigma \epsilon_\sigma q^{r+s} \cdot L^{\varphi,-}_{\sigma} \otimes L^{\varphi,+}_{\sigma} \tag{5.8}$$

(notice that $\widehat{U}^Q_{q,\varphi}(\mathfrak{t}) = \widehat{U}^Q_q(\mathfrak{t})$ as Hopf algebras). Now if $f \in \widehat{A}^{\varphi,P}_{\infty}$, we have that $\left(\widehat{F}_{\bar{\sigma}} \triangleright f \triangleleft \widehat{E}_{\bar{\sigma}} \right) \Big|_{\widehat{U}^Q_{q,\varphi}(\mathfrak{t})^{\otimes 2}}$ is an integer valued function on $\widehat{U}^Q_{q,\varphi}(\mathfrak{t})^{\otimes 2} = \widehat{U}^Q_q(\mathfrak{t})^{\otimes 2}$, whence from (5.8) and (2.3) we get $a_\sigma \epsilon_\sigma q^{r+s} \in k[q, q^{-1}]$, hence $a_\sigma \in k[q, q^{-1}]$, for all $\sigma \in \mathcal{S}$; in particular $a_{\bar{\sigma}} \in k[q, q^{-1}]$: since this can be done for all σ such that $a_\sigma \neq 0$ we conclude that $a_\sigma \in k[q, q^{-1}]$ for all coefficients a_σ. Conversely, if the latter holds we clearly have $f \in \widehat{A}^{\varphi,P}_{\infty}$. It is then clear that $\widehat{A}^{\varphi,P}_{\infty}$ has the above PBW pseudobasis over $k[q, q^{-1}]$, has the claimed set of (topological) generators, and is a $k[q, q^{-1}]$-form of $A^{\varphi,P}_{\infty}$.

Now look at the Hopf structure. Let $f \in \widehat{A}^{\varphi,P}_{\infty}$; write $\Delta(f)$ in terms of the PBW basis,

$$\Delta(f) = \sum_{\sigma} a_\sigma \cdot \left(\overline{F}^{\varphi}_{\sigma} L^{\varphi,-}_{\sigma} \otimes L^{\varphi,+}_{\sigma} \overline{E}^{\varphi}_{\sigma} \right) \otimes \left(\left(\overline{F}^{\varphi}_{\sigma} \right)' \left(L^{\varphi,-}_{\sigma} \right)' \otimes \left(L^{\varphi,+}_{\sigma} \right)' \left(\overline{E}^{\varphi}_{\sigma} \right)' \right)$$

(for possibly infinite σ); as above we have to prove that $a_\sigma \in k[q, q^{-1}]$ for all σ, and we proceed as in the first part of the proof. Since $f \in \widehat{A}^{\varphi,P}_{\infty}$, i. e. it is an integer valued function on $\widehat{U}^Q_{q,\varphi}(\mathfrak{g})$, then $\Delta(f)$ is an integer valued function on $\widehat{U}^Q_{q,\varphi}(\mathfrak{g}) \otimes \widehat{U}^Q_{q,\varphi}(\mathfrak{g})$. For any fixed $\bar{\sigma}$ such that $a_{\bar{\sigma}} \neq 0$ we locate PBW monomials $\widehat{E}_{\bar{\sigma}}$, $\widehat{F}_{\bar{\sigma}}$, $\widehat{E}'_{\bar{\sigma}}$, $\widehat{F}'_{\bar{\sigma}}$ such that

$$\left(\left(\widehat{F}^{\varphi}_{\bar{\sigma}} \otimes \widehat{F}'_{\bar{\sigma}} \right) \triangleright \Delta(f) \triangleleft \left(\widehat{E}_{\bar{\sigma}} \otimes \widehat{E}'_{\bar{\sigma}} \right) \right) \Big|_{\widehat{U}^Q_{q,\varphi}(\mathfrak{t})^{\otimes 2}} =$$
$$= \sum_{\sigma \in \mathcal{S}} a_\sigma \epsilon_\sigma q^{r+s+r'+s'} \cdot \left(L^{\varphi,-}_{\sigma} \otimes L^{\varphi,+}_{\sigma} \right) \otimes \left(\left(L^{\varphi,-}_{\sigma} \right)' \otimes \left(L^{\varphi,+}_{\sigma} \right)' \right)$$

where $\mathcal{S} := \left\{ \sigma \,\middle|\, \overline{F}^{\varphi}_{\sigma} = \overline{F}^{\varphi}_{\bar{\sigma}}, \overline{E}^{\varphi}_{\sigma} = \overline{E}^{\varphi}_{\bar{\sigma}}, \left(\overline{F}^{\varphi}_{\sigma} \right)' = \left(\overline{F}^{\varphi}_{\bar{\sigma}} \right)', \left(\overline{E}^{\varphi}_{\sigma} \right)' = \left(\overline{E}^{\varphi}_{\bar{\sigma}} \right)' \right\}$ and $\epsilon_\sigma = \pm 1$; but $\left(\left(\widehat{F}_{\bar{\sigma}} \otimes \widehat{F}'_{\bar{\sigma}} \right) \triangleright \Delta(f) \triangleleft \left(\widehat{E}_{\bar{\sigma}} \otimes \widehat{E}_{\bar{\sigma}} \right) \right) \Big|_{\widehat{U}^Q_q(\mathfrak{t})^{\otimes 2}}$ is integer valued, hence like above we get $a_\sigma \epsilon_\sigma q^{r+s+r'+s'} \in k[q, q^{-1}]$ and $a_\sigma \in k[q, q^{-1}]$ for all $\sigma \in \mathcal{S}$, whence we conclude the same for all a_σ: thus $\Delta(f) \in \widehat{A}^{\varphi,P}_{\infty} \widehat{\otimes} \widehat{A}^{\varphi,P}_{\infty}$, i. e. $\widehat{A}^{\varphi,P}_{\infty}$ is a (formal) subcoalgebra of $A^{\varphi,P}_{\infty}$. Furthermore, $1 \in \widehat{A}^{\varphi,P}_{\infty}$ by definition, $\varepsilon \left(\widehat{A}^{\varphi,P}_{\infty} \right) \leq k[q, q^{-1}]$, because $1 \in \widehat{U}^Q_{q,\varphi}(\mathfrak{g})$

and $S\left(\widehat{A}_\infty^{\varphi,P}\right) = \widehat{A}_\infty^{\varphi,P}$ because $S\left(\widehat{U}_{q,\varphi}^{Q}(\mathfrak{g})\right) = \widehat{U}_{q,\varphi}^{Q}(\mathfrak{g})$. Thus $\widehat{A}_\infty^{\varphi,P}$ is a (formal) Hopf subalgebra of $A_\infty^{\varphi,P}$ over $k[q,q^{-1}]$, q.e.d. The last part of claim *(a)* is trivial: just remark that (for all $\lambda \in P_+$) $L_{(1+\varphi)(\lambda)} \otimes L_{-(1-\varphi)(\lambda)} = \left(L_{-(1+\varphi)(\lambda)} \otimes L_{(1-\varphi)(\lambda)}\right)^{-1} = \left(1-\left(1-L_{-(1+\varphi)(\lambda)} \otimes L_{(1-\varphi)(\lambda)}\right)\right)^{-1} = \sum_{n=0}^{\infty}\left(1-L_{-(1+\varphi)(\lambda)} \otimes L_{(1-\varphi)(\lambda)}\right)^{n}$.

(b) mimick step by step the proof of *(a)*. $\square$

5.10 *Remarks:* *(a)* Notice that, letting

$$\widehat{A}^{\varphi,P} := \left\{ f \in A^{\varphi,P} \,\middle|\, \left\langle f, \widehat{U}_{q,\varphi}^{Q}(\mathfrak{g}) \right\rangle \subseteq k[q,q^{-1}] \right\}$$
$$\widetilde{A}^{\varphi,Q} := \left\{ f \in A^{\varphi,Q} \,\middle|\, \left\langle f, \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) \right\rangle \subseteq k[q,q^{-1}] \right\}$$

one can define the *algebra* $\widehat{A}_\infty^{\varphi,P}$, resp. $\widetilde{A}_\infty^{\varphi,Q}$, as the $\mathfrak{E}'$-adic completion of the subalgebra $\widehat{A}^{\varphi,P}$, resp. $\widetilde{A}^{\varphi,Q}$, where $\mathfrak{E}' := \mathfrak{E} \cap \widehat{A}^{\varphi,P}$ with $\mathfrak{E} := Ker\left(\varepsilon\colon A_\infty^{\varphi,P} \to k(q)\right)$, resp. $\mathfrak{E}' := \mathfrak{E} \cap \widetilde{A}^{\varphi,Q}$ with $\mathfrak{E} := Ker\left(\varepsilon\colon A_\infty^{\varphi,Q} \to k(q)\right)$); therefore $\widehat{A}_\infty^{\varphi,P}$, resp. $\widetilde{A}_\infty^{\varphi,Q}$, is a formal Hopf algebra over $k[q,q^{-1}]$, complete in the $\mathfrak{E}''$-adic topology, with $\mathfrak{E}'' := Ker\left(\varepsilon\colon \widehat{A}_\infty^{\varphi,P} \to k[q,q^{-1}]\right)$, resp. $\mathfrak{E}'' := Ker\left(\varepsilon\colon \widetilde{A}_\infty^{\varphi,Q} \to k[q,q^{-1}]\right)$.

(b) The analysis in the proof of Proposition 5.9 also shows that

$$\widehat{U}_{q,\varphi}^{Q}(\mathfrak{g}) := \left\{ x \in U_{q,\varphi}^{Q}(\mathfrak{g}) \,\middle|\, \left\langle \widehat{A}^{\varphi,P}, x \right\rangle \subseteq k[q,q^{-1}] \right\}$$
$$\widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) := \left\{ x \in U_{q,\varphi}^{P}(\mathfrak{g}) \,\middle|\, \left\langle \widetilde{A}^{\varphi,Q}, x \right\rangle \subseteq k[q,q^{-1}] \right\}.$$

The following proposition — whose proof is trivial from the previous results — describes the behavior of integer forms under the natural embeddings $\mu^{\varphi,M}$, $\mu_\infty^{\varphi,M}$; in particular, it affords another proof of the fact that $\widehat{F}_{q,\varphi}^{P}[G]$, resp. $\widetilde{F}_{q,\varphi}^{Q}[G]$, is a Hopf subalgebra (over $k[q,q^{-1}]$) of $F_{q,\varphi}^{P}[G]$, resp. $F_{q,\varphi}^{Q}[G]$.

Proposition 5.11. *The natural embeddings preserve integer forms, namely*

$$\widehat{F}_{q,\varphi}^{P}[G] = \left(\mu^{\varphi,P}\right)^{-1}\left(\widehat{A}_\infty^{\varphi,P}\right), \quad \widetilde{F}_{q,\varphi}^{Q}[G] = \left(\mu^{\varphi,Q}\right)^{-1}\left(\widetilde{A}_\infty^{\varphi,Q}\right)$$
$$\widehat{F}_{q,\varphi}^{P,\infty}[G] = \left(\mu_\infty^{\varphi,P}\right)^{-1}\left(\widehat{A}_\infty^{\varphi,P}\right), \quad \widetilde{F}_{q,\varphi}^{Q,\infty}[G] = \left(\mu_\infty^{\varphi,Q}\right)^{-1}\left(\widetilde{A}_\infty^{\varphi,Q}\right)$$

so that restriction gives embeddings of formal Hopf algebras over $k[q,q^{-1}]$

$$\mu^{\varphi,P}\colon \widehat{F}_{q,\varphi}^{P}[G] \hookrightarrow \widehat{A}_\infty^{\varphi,P}, \quad \mu^{\varphi,Q}\colon \widetilde{F}_{q,\varphi}^{Q}[G] \hookrightarrow \widetilde{A}_\infty^{\varphi,Q}$$
$$\mu_\infty^{\varphi,P}\colon \widehat{F}_{q,\varphi}^{P,\infty}[G] \hookrightarrow \widehat{A}_\infty^{\varphi,P}, \quad \mu_\infty^{\varphi,Q} \widetilde{F}_{q,\varphi}^{Q,\infty}[G] \hookrightarrow \widetilde{A}_\infty^{\varphi,Q}. \ \square$$

5.12. The result above can in fact be refined, extending embeddings to isomorphisms. Let $\mu \in M \cap P_+$, and let $V_{-\mu}$ be an irreducible $U_{q,\varphi}^{M'}(\mathfrak{g})$-module of lowest weight $-\mu$ (recall that $U_{q,\varphi}^{M'}(\mathfrak{g}) \cong U_q^{M'}(\mathfrak{g})$ as algebras, hence their representation theory is the same). Let $v_{-\mu} \neq 0$ be a lowest weight vector of $V_{-\mu}$, and let $\phi_{-\mu} \in {V_{-\mu}}^*$ be the linear functional on $V_{-\mu}$ defined by *(a)* $\phi_{-\mu}(v_{-\mu}) = 1$ and *(b)* $\phi_{-\mu}$ vanishes on the unique $U_{q,\varphi}^{M'}(\mathfrak{t})$-invariant complement of $k(q).v_{-\mu}$ in $V_{-\mu}$; let $\psi_{-\mu} := c_{\phi_{-\mu},v_{-\mu}}$ be the corresponding matrix coefficient, i. e. $\psi_{-\mu}\colon x \mapsto \phi_{-\mu}(x.v_{-\mu})$ for all $x \in U_{q,\varphi}^{M'}(\mathfrak{g})$; finally, let $\rho := \sum_{i=1}^{n} \omega_i$,

$\delta := \sum_{i=1}^n \alpha_i$. Then we can refine Proposition 5.11 with the following result, which improves [DL], Theorem 4.6, and [CV-2], Lemma 2.5:

Theorem 5.13.
(a1) The embedding $\mu^{\varphi,P} \colon \widehat{F}_{q,\varphi}^P[G] \hookrightarrow \widehat{A}^{\varphi,P}$ *extends to a* $k[q,q^{-1}]$*-algebra isomorphism*

$$\mu^{\varphi,P} \colon \widehat{F}_{q,\varphi}^P[G]\left[\psi_{-\rho}^{-1}\right] \xrightarrow{\cong} \widehat{A}^{\varphi,P} ;$$

it follows that $\widehat{F}_{q,\varphi}^P[G]$ *is a* $k[q,q^{-1}]$*-form of* $F_{q,\varphi}^P[G]$*.*

(a2) The algebra $\widehat{F}_{q,\varphi}^P[G]\left[\psi_{-\rho}^{-1}\right]$ *is naturally embedded into* $\widehat{F}_{q,\varphi}^{P,\infty}[G]$*; it follows that the isomorphism in (a1) coincides with the restriction of* $\mu_\infty^{\varphi,P}$*, and the latter is in fact an isomorphism* $\mu_\infty^{\varphi,P} \colon \widehat{F}_{q,\varphi}^{P,\infty}[G] \xrightarrow{\cong} \widehat{A}_\infty^{\varphi,P}$ *of formal Hopf algebras over* $k[q,q^{-1}]$*.*

(b1) The embedding $\mu^{\varphi,Q} \colon \widetilde{F}_{q,\varphi}^Q[G] \hookrightarrow \widetilde{A}^{\varphi,Q}$ *extends to a* $k[q,q^{-1}]$*-algebra isomorphism*

$$\mu^{\varphi,Q} \colon \widetilde{F}_{q,\varphi}^Q[G]\left[\psi_{-\delta}^{-1}\right] \xrightarrow{\cong} \widetilde{A}^{\varphi,Q} ;$$

it follows that $\widetilde{F}_{q,\varphi}^Q[G]$ *is a* $k[q,q^{-1}]$*-form of* $F_{q,\varphi}^Q[G]$*.*

(b2) The algebra $\widetilde{F}_{q,\varphi}^Q[G]\left[\psi_{-\delta}^{-1}\right]$ *is naturally embedded into* $\widetilde{F}_{q,\varphi}^{Q,\infty}[G]$*; it follows that the isomorphism in (b1) coincides with the restriction of* $\mu_\infty^{\varphi,Q}$*, and the latter is in fact an isomorphism* $\mu_\infty^{\varphi,Q} \colon \widetilde{F}_{q,\varphi}^{Q,\infty}[G] \xrightarrow{\cong} \widetilde{A}_\infty^{\varphi,Q}$ *of formal Hopf algebras over* $k[q,q^{-1}]$*.*

Proof. Once again we stick for simplicity to the case $\varphi = 0$.

It is shown in [DL], §§4.4–6, that $\mu^P \colon \widehat{F}_q^P[G] \hookrightarrow \widehat{A}^P$ extends to a $k[q,q^{-1}]$-algebra isomorphism $\mu^P \colon \widehat{F}_q^P[G]\left[\psi_{-\rho}^{-1}\right] \cong \widehat{A}^P$; then scalar extension gives a $k(q)$-algebra isomorphism $\mu^P \colon F_q^P[G]\left[\psi_{-\rho}^{-1}\right] \cong A^P$; then $\widehat{A}^P$ be a $k[q,q^{-1}]$-form of A^P implies $\widehat{F}_{q,\varphi}^P[G]$ be a $k[q,q^{-1}]$-form of $F_{q,\varphi}^P[G]$; thus *(a1)* holds. The same argument (disregarding integrality) shows that $\mu^Q \colon F_q^Q[G] \hookrightarrow A^Q$ extends to a $k(q)$-algebra isomorphism $\mu^Q \colon F_q^Q[G]\left[\psi_{-\delta}^{-1}\right] \cong A^Q$. Furthermore, computations in [DL] give $\mu^Q(\psi_{-\mu}) = K_{-\mu} \otimes K_\mu$ for all weights μ ; thus in particular for $\mu = \alpha_i$ and $\mu = \delta$ we get

$$\mu^Q\left(\psi_{-\alpha_i}\right) = K_i^{-1} \otimes K_i \, , \; \mu^Q\left(\psi_{-\delta}\right) = K_{-\delta} \otimes K_\delta \, , \; \mu^Q\left(\psi_{-\delta}^{-1}\right) = K_\delta \otimes K_{-\delta} \tag{5.9}$$

whence $K_i^{-1} \otimes K_i \in Im\left(\mu^Q\right)$ for all $i = 1, \dots, n$. Even more, from the proof of Theorem 4.6 of [DL] we also get that $F_i \otimes 1, 1 \otimes E_i \in \mu^Q\left(F_q^Q[G]\right)$, hence also $F_i^{(f)} \otimes 1, 1 \otimes E_i^{(e)} \in \mu^Q\left(F_q^Q[G]\right)$ (for all $i = 1, \dots, n$, $f, e \in \mathbb{N}$); then from Proposition 5.11 we get $F_i^{(f)} \otimes 1, 1 \otimes E_i^{(e)} \in \mu^Q\left(\widetilde{F}_q^Q[G]\right)$, for all $i = 1, \dots, n$ and $f, e \in \mathbb{N}$; similarly, we have $K_i^{-1} \otimes K_i = \mu^Q(z_i) \in \mu^Q\left(\widetilde{F}_q^Q[G]\right)$, with $z_i := \psi_{-\alpha_i} \in \widetilde{F}_q^Q[G]$, because of Proposition 5.11. Thus (from (5.9))

$$K_i \otimes K_i^{-1} = \left(\prod_{\substack{j=1 \\ j \neq i}}^n K_j^{-1} \otimes K_j \right) \cdot \left(K_\delta \otimes K_{-\delta}\right) \in \mu^Q\left(\widetilde{F}_q^Q[G]\left[\psi_{-\delta}^{-1}\right]\right)$$

and on the other hand

$$\binom{K_i^{-1} \otimes K_i; c}{t} := \prod_{s=1}^{t} \frac{q_i^{c-s+1} \cdot K_i^{-1} \otimes K_i - 1}{q_i^s - 1} =$$

$$= \prod_{s=1}^{t} \frac{\mu^Q(z_i) q_i^{c-s+1} - 1}{q_i^s - 1} = \binom{\mu^Q(z_i); c}{t} = \mu^Q\left(\binom{z_i; c}{t}\right)$$

and again by Proposition 5.11 we get $\binom{z_i;c}{t} \in \widetilde{F}_q^Q[G]$, thus $\binom{K_i^{-1} \otimes K_i;c}{t}$ (for all i, c, t). All this and Proposition 5.9 give $\mu^Q\left(\widetilde{F}_q^Q[G]\left[\psi_{-\delta}^{-1}\right]\right) = \widetilde{A}^Q$; then $\widetilde{A}^Q$ be a $k[q,q^{-1}]$-form of A^Q implies $\widetilde{F}_{q,\varphi}^Q[G]$ be a $k[q,q^{-1}]$-form of $F_{q,\varphi}^Q[G]$, thus *(b1)* holds.

Now consider *(a1)*: notice that $K_{-\delta} \otimes K_\delta = 1 - (1 - K_{-\delta} \otimes K_\delta) = 1 - y$, where $y := 1 - K_{-\delta} \otimes K_\delta \in \mathfrak{E} := Ker(\varepsilon \colon \widehat{A}_\infty^P \to k[q,q^{-1}])$; thus on the other side we have $\psi_{-\delta} = 1 - (1 - \psi_{-\delta}) = 1 - x$, where $x := 1 - \psi_{-\delta} \in \mathfrak{E} := Ker\left(\varepsilon \colon \widehat{F}_q^P[G] \to k[q,q^{-1}]\right)$ (for $\mu^P \colon \widehat{F}_q^P[G] \hookrightarrow \widehat{A}_\infty^P$ is a morphism of formal Hopf algebras); therefore $\psi_{-\delta}$ is invertible in $\widehat{F}_q^{P,\infty}[G]$, its inverse being $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (1 - \psi_{-\delta})^n$ (cf. §4.5), hence $\widehat{F}_q^P[G]\left[\psi_{-\delta}^{-1}\right]$ embeds into $\widehat{F}_q^{P,\infty}[G]$ and *(a2)* follows. Similarly for *(b2)*.

The same argument works for general φ too, starting from [CV-2], Proposition 2.7. □

5.14 Gradings. Recall that $U_{q,\varphi}^M(\mathfrak{b}_\pm)$ has a $Q_\pm$-grading $U_{q,\varphi}^M(\mathfrak{b}_\pm) = \oplus_{\alpha \in Q_\pm} \left(U_{q,\varphi}^M(\mathfrak{b}_\pm)\right)_\alpha$ given by the direct sum decomposition into weight spaces for the adjoint action of $U_{q,\varphi}^M(\mathfrak{t})$. These are gradings of Hopf algebras (in the obvious usual sense), which are inherited by integer forms $\widehat{U}_{q,\varphi}^Q(\mathfrak{b}_+)$, $\widetilde{U}_{q,\varphi}^P(\mathfrak{b}_-)$, etc. etc., and are compatible with (or respected by) DRT pairings, viz. for instance $\pi_-^\varphi\left(\left(U_{q,\varphi}^Q(\mathfrak{b}_-)\right)_\beta, \left(U_{q,\varphi}^P(\mathfrak{b}_+)\right)_\gamma\right) = 0$ for all $\beta \in Q_-$, $\gamma \in Q_+$ such that $\beta + \gamma \neq 0$.

The gradings of Borel subalgebras provide a Q-grading of the Hopf algebra $D_{q,\varphi}^M(\mathfrak{g}) = U_{q,\varphi}^M(\mathfrak{b}_+) \otimes U_{q,\varphi}^Q(\mathfrak{b}_-)$ (inherited by the quotient Hopf algebra $U_{q,\varphi}^M(\mathfrak{g})$), the subspace $\left(U_{q,\varphi}^M(\mathfrak{b}_+)\right)_\beta \otimes \left(U_{q,\varphi}^Q(\mathfrak{b}_-)\right)_\gamma$ having degree $\beta - \gamma$, and a Q-grading of the formal Hopf algebra $U_{q,\varphi}^M(\mathfrak{b}_+) \widehat{\otimes} U_{q,\varphi}^Q(\mathfrak{b}_-)$; it is clear that the restriction of the natural evaluation pairing $D_{q,\varphi}^M(\mathfrak{g})^* \otimes D_{q,\varphi}^M(\mathfrak{g}) \to k(q)$ respects these gradings.

5.15 Computational results. We now look for concrete information about the formal Hopf algebra structure of $A_\infty^{\varphi,P}$ and $A_\infty^{\varphi,Q}$ and their integer forms.

First of all, the counity $\varepsilon \colon D_{q,\varphi}^Q(\mathfrak{g})^* \to k(q)$ is defined as $\varepsilon := 1^*$, hence $\varepsilon(x^*) := \langle x^*, 1 \rangle$, for all $x^* \in D_{q,\varphi}^Q(\mathfrak{g})^*$; thus in particular the restriction $\varepsilon \colon \widehat{A}_\infty^{\varphi,P} \to k[q,q^{-1}]$ is defined by

$$\varepsilon\left(\overline{F}_{\alpha^k}^\varphi \otimes 1\right) = 0 \,, \quad \varepsilon\left(L_{-(1+\varphi)(\lambda)} \otimes L_{(1-\varphi)(\lambda)}\right) = 1 \,, \quad \varepsilon\left(1 \otimes \overline{E}_{\alpha^k}^\varphi\right) = 0$$

and by scalar extension these same formulas define also $\varepsilon \colon A_\infty^{\varphi,P} \to k(q)$. Similarly,

$\varepsilon \colon \widetilde{A}_\infty^{\varphi,Q} \to k[q,q^{-1}]$ comes out to be defined by

$$\varepsilon\Big((F_i^\varphi)^{(f)} \otimes 1\Big) = 0\,, \qquad \varepsilon\left(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\right) = 1$$
$$\varepsilon\left(\begin{pmatrix} K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}; c \\ t \end{pmatrix}\right) = \begin{pmatrix} c \\ t \end{pmatrix}_{q_i}\,,$$
$$\varepsilon\left(K_i \otimes K_i^{-1}\right) = 1\,, \qquad \varepsilon\Big(1 \otimes (E_i^\varphi)^{(e)}\Big) = 0$$

and by scalar extension these same formulas define also $\varepsilon \colon A_\infty^{\varphi,Q} \to k(q)$.

Now for the antipode $S \colon D_{q,\varphi}^Q(\mathfrak{g})^* \to D_{q,\varphi}^P(\mathfrak{g})^*$: it is by definition the transpose of the antipode of $D_{q,\varphi}^Q(\mathfrak{g})$, hence it is characterized by $\langle S(x^*), x \rangle = \langle x^*, S(x) \rangle$, for all $x^* \in D_{q,\varphi}^Q(\mathfrak{g})^*$, $x \in D_{q,\varphi}^Q(\mathfrak{g})$. Now consider $F_i^\varphi \otimes 1 \in U_{q,\varphi}^P(\mathfrak{b}_-) \,\widehat{\otimes}\, U_{q,\varphi}^P(\mathfrak{b}_+) \leq D_{q,\varphi}^Q(\mathfrak{g})^*$: it is homogeneous of degree $-\alpha_i$, whence $S\,(F_i^\varphi \otimes 1)$ has the same degree (for S is degree preserving, the Q-grading being compatible with the (formal) Hopf structure: cf. §5.14). Thus writing $S\,(F_i^\varphi \otimes 1)$ as a series

$$S\,(F_i^\varphi \otimes 1) = \textstyle\sum_\sigma c_\sigma \cdot F_\sigma^\varphi L_\sigma^{\varphi,-} \otimes L_\sigma^{\varphi,+} E_\sigma^\varphi$$

(where $F_\sigma^\varphi L_\sigma^{\varphi,-} \otimes L_\sigma^{\varphi,+} E_\sigma^\varphi$ denotes a PBW monomial) we have $c_\sigma \neq 0$ only if $deg(F_\sigma^\varphi L_\sigma^{\varphi,-} \otimes L_\sigma^{\varphi,+} E_\sigma^\varphi) := deg(F_\sigma^\varphi) + deg(E_\sigma^\varphi) = -\alpha_i$.

The integer form $\widehat{A}_\infty^{\varphi,P}$ of $A_\infty^{\varphi,P}$ inherits a Q-grading, and the same procedure gives us

$$S\left(\overline{F}_{\alpha^h}^\varphi \otimes 1\right) = \sum_\sigma c_\sigma^- \cdot \overline{F}_\sigma^\varphi L_\sigma^{\varphi,-} \otimes L_\sigma^{\varphi,+} \overline{E}_\sigma^\varphi \tag{5.10}$$

(for $h = 1, \dots, N$) where $c_\sigma^- \in k[q,q^{-1}]$ and the $\overline{F}_\sigma^\varphi L_\sigma^{\varphi,-} \otimes L_\sigma^{\varphi,+} \overline{E}_\sigma^\varphi$'s are PBW monomials of degree $deg(\overline{F}_\sigma^\varphi) + deg(\overline{E}_\sigma^\varphi) = -\alpha^h$. Similarly for $(F_i^\varphi)^{(f)} \otimes 1 \in \widetilde{A}_\infty^{\varphi,Q}$ one has

$$S\left((F_i^\varphi)^{(f)} \otimes 1\right) = \sum_\sigma a_\sigma^- \cdot \widehat{F}_\sigma^\varphi (K_\sigma^{\varphi,-}) \otimes (K_\sigma^{\varphi,+}) \widehat{E}_\sigma^\varphi \tag{5.11}$$

(for $f \in \mathbb{N}, i = 1, \dots, n$) where $a_\sigma^- \in k[q,q^{-1}]$ and the $\widehat{F}_\sigma^\varphi (K_\sigma^{\varphi,-}) \otimes (K_\sigma^{\varphi,+}) \widehat{E}_\sigma^\varphi$'s are PBW monomials of degree $deg(\widehat{F}_\sigma^\varphi) + deg(\widehat{E}_\sigma^\varphi) = -f\alpha_i$. But we can compare $\widehat{A}_\infty^{\varphi,P}$ and $\widetilde{A}_\infty^{\varphi,Q}$ by means of the obvious embedding $A_\infty^{\varphi,Q} \hookrightarrow A_\infty^{\varphi,P}$; since $\big(\overline{F}_{\alpha^h}^\varphi\big)^f \otimes 1 = \big(q_{\alpha^h} - q_{\alpha^h}^{-1}\big)^f \cdot [f]_{q_{\alpha^h}} \cdot (F_{\alpha^h}^\varphi)^{(f)} \otimes 1$ and $1 \otimes \big(\overline{E}_{\alpha^k}^\varphi\big)^e = \big(q_{\alpha^k} - q_{\alpha^k}^{-1}\big)^e \cdot [e]_{q_{\alpha^k}} \cdot 1 \otimes (E_{\alpha^k}^\varphi)^{(e)}$ ($h, k = 1, \dots, N; f, e \in \mathbb{N}$), direct comparison between (5.10) and (5.11) gives $a_\sigma^- = \big(q - q^{-1}\big)^{\mathrm{e}_\sigma} \cdot b_\sigma \cdot c_\sigma^-$, where $b_\sigma \in k[q,q^{-1}]$ and $\mathrm{e}_\sigma := \sum_{s=1}^N (f_s + e_s)$, with $\widehat{F}_\sigma^\varphi = \prod_{h=N}^1 (F_{\alpha^h}^\varphi)^{(f_h)}$, $\widehat{E}_\sigma^\varphi = \prod_{k=1}^N (E_{\alpha^k}^\varphi)^{(e_k)}$. In particular, we can write $S\Big((F_i^\varphi)^{(f)} \otimes 1\Big)$ as a series

$$S\left((F_i^\varphi)^{(f)} \otimes 1\right) = \sum_{\kappa=0}^\infty \big(q - q^{-1}\big)^\kappa \cdot d_\kappa \cdot \widehat{F}_\kappa^\varphi (K_\kappa^{\varphi,-}) \otimes (K_\kappa^{\varphi,+}) \widehat{E}_\kappa^\varphi \tag{5.12}$$

(with $d_\kappa \in k[q,q^{-1}]$ for all $\kappa \in \mathbb{N}$) which is convergent in the $(q-1)$-adic topology of $\widetilde{A}_\infty^{\varphi,Q}$.

Similarly one finds that $S\left(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\right)$, $S\left(K_{-(1+\varphi)(-\alpha_i)} \otimes K_{(1-\varphi)(-\alpha_i)}\right)$, $S\left(\left({K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)};c \atop t}\right)\right)$ and $S\left(1 \otimes \left(E_i^\varphi\right)^{(e)}\right)$ are expressed by series which are convergent in the $(q-1)$-adic topology of $\widetilde{A}_\infty^{\varphi,Q}$. In particular, all such series are in fact *finite* sums modulo $(q-1)$: we will now compute the first terms of these series.

For $S\left(F_i^\varphi \otimes 1\right)$ the first term — call it $\mathcal{F}_1$ — attached to $\kappa = 0$ in the series (5.12) corresponds to the summands $a_\sigma^- \cdot \widehat{F}_\sigma^\varphi(K_\sigma^{\varphi,-}) \otimes (K_\sigma^{\varphi,+})\widehat{E}_\sigma^\varphi$ in (5.11) such that $\partial\left(\widehat{F}_\sigma^\varphi\right) + {}+\partial\left(\widehat{E}_\sigma^\varphi\right) = 1$, with $\partial\left(\prod_{h=1}^N \left(F_{\alpha^h}^\varphi\right)^{(f_h)}\right) := \sum_{h=N}^1 f_h$, $\partial\left(\prod_{k=N}^1 \left(E_{\alpha^k}^\varphi\right)^{(e_k)}\right) := \sum_{k=1}^N e_k$; thus we find a condition $\sum_{s=1}^N (f_s + e_s) = 1$; on the other hand, these terms must have degree $deg\left(\widehat{F}_\sigma^\varphi\right) + deg\left(\widehat{E}_\sigma^\varphi\right) = -\alpha_i$: we conclude that $\widehat{F}_\sigma^\varphi = F_i^\varphi$ and $\widehat{E}_\sigma^\varphi = 1$. Now remark that $\mathcal{F}_1$ takes non-zero values only on PBW monomials (of $\widetilde{U}_{q,\varphi}^P(\mathfrak{g})$) like $\overline{E}_i^\varphi \cdot L_\lambda$, $\lambda \in P$; let $V_{1,i}$ be the $k[q,q^{-1}]$-span of such monomials: this is a free $U_{q,\varphi}^P(\mathfrak{t})$-module of rank 1; then direct computation shows that $\mathcal{F}_1 + q_i^{-2} \cdot F_i^\varphi K_{-(1+\varphi)(-\alpha_i)} \otimes K_{(1-\varphi)(-\alpha_i)}$ is zero in $(V_{1,i})^*$: we conclude that $\mathcal{F}_1 = -q_i^{-2} \cdot F_i^\varphi K_{-(1+\varphi)(-\alpha_i)} \otimes K_{(1-\varphi)(-\alpha_i)}$, hence we can write (for all $i = 1, \dots, n$)

$$S\left(F_i^\varphi \otimes 1\right) \equiv -q^{(\alpha_i|\alpha_i+\tau_i)} \cdot F_i^\varphi K_{-(1+\varphi)(-\alpha_i)} \otimes K_{(1-\varphi)(-\alpha_i)} \qquad \mod (q-1)^2 .$$

With similar arguments we find that (for all $i = 1, \dots, n$, $c \in \mathbb{Z}$, $t \in \mathbb{N}$)

$$\begin{aligned} &S\left(K_{-(1+\varphi)(\pm\alpha_i)} \otimes K_{(1-\varphi)(\pm\alpha_i)}\right) \equiv K_{-(1+\varphi)(\mp\alpha_i)} \otimes K_{(1-\varphi)(\mp\alpha_i)} && \mod (q-1)^2 \\ &S\left(\left({K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)};c \atop t}\right)\right) \equiv -K_{-(1+\varphi)(-\alpha_i)} \otimes K_{(1-\varphi)(-\alpha_i)} \cdot \\ &\qquad \cdot \left({K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)});c \atop t}\right) && \mod (q-1)^2 \\ &S(1 \otimes E_i^\varphi) \equiv -q^{(\alpha_i|\alpha_i-\tau_i)} \cdot K_{-(1+\varphi)(-\alpha_i)} \otimes K_{(1-\varphi)(-\alpha_i)} E_i^\varphi && \mod (q-1)^2 \end{aligned}$$

The comultiplication $\Delta\colon D_{q,\varphi}^Q(\mathfrak{g})^* \to D_{q,\varphi}^Q(\mathfrak{g})^* \mathbin{\widehat{\otimes}} D_{q,\varphi}^Q(\mathfrak{g})^*$ is by definition the transpose of the multiplication of $D_{q,\varphi}^Q(\mathfrak{g})$, hence it is characterized by $\langle \Delta(x^*), y \otimes z \rangle = \langle x^*, y \cdot z \rangle$, for all $x^* \in D_{q,\varphi}^Q(\mathfrak{g})^*$, $y, z \in D_{q,\varphi}^Q(\mathfrak{g})$. Now consider $F_i^\varphi \otimes 1 \in U_{q,\varphi}^P(\mathfrak{b}_-) \mathbin{\widehat{\otimes}} U_{q,\varphi}^P(\mathfrak{b}_+) \leq D_{q,\varphi}^Q(\mathfrak{g})^*$: it is homogeneous of degree $-\alpha_i$, whence $\Delta\left(F_i^\varphi \otimes 1\right)$ has the same degree. Thus writing $\Delta\left(F_i^\varphi \otimes 1\right)$ as a series

$$\Delta\left(F_i^\varphi \otimes 1\right) = \textstyle\sum_\sigma s_\sigma \cdot \left(F_\sigma^\varphi L_\sigma^{\varphi,-} \otimes L_\sigma^{\varphi,+} E_\sigma^\varphi\right) \otimes \left(F_\sigma^{\varphi\prime} L_\sigma^{\varphi,-\prime} \otimes L_\sigma^{\varphi,+\prime} E_\sigma^{\varphi\prime}\right)$$

(where $F_\sigma^\varphi L_\sigma^{\varphi,-} \otimes L_\sigma^{\varphi,+} E_\sigma^\varphi$ and $F_\sigma^{\varphi\prime} L_\sigma^{\varphi,-\prime} \otimes L_\sigma^{\varphi,+\prime} E_\sigma^{\varphi\prime}$ are PBW monomials) we have $c_\sigma \neq {}\neq 0$ only if $deg((F_\sigma^\varphi L_\sigma^{\varphi,-} \otimes L_\sigma^{\varphi,+} E_\sigma^\varphi) \otimes (F_\sigma^{\varphi\prime} L_\sigma^{\varphi,-\prime} \otimes L_\sigma^{\varphi,+\prime} E_\sigma^{\varphi\prime})) := deg(F_\sigma^\varphi) + deg(E_\sigma^\varphi) + {}+deg(F_\sigma^{\varphi\prime}) + deg(E_\sigma^{\varphi\prime}) = -\alpha_i$. Applying this argument to $\widehat{A}_\infty^{\varphi,P}$ and proceeding like for the antipode, we get

$$\Delta\left(\overline{F}_{\alpha_i}^\varphi \otimes 1\right) = \sum_\sigma c_\sigma^- \cdot \left(\overline{F}_\sigma^\varphi L_\sigma^{\varphi,-} \otimes L_\sigma^{\varphi,+} \overline{E}_\sigma^\varphi\right) \otimes \left(\left(\overline{F}_\sigma^\varphi\right)' (L_\sigma^{\varphi,-})' \otimes (L_\sigma^{\varphi,+})' \left(\overline{E}_\sigma^\varphi\right)'\right) \quad (5.13)$$

where $c_\sigma^- \in k[q,q^{-1}]$ and the occurring PBW monomials have degree $-\alpha_i$. Similarly

$$\Delta\Big(\big(F_i^\varphi\big)^{(f)} \otimes 1\Big) = \sum_\sigma a_\sigma^- \cdot \Big(\widehat{F}_\sigma^\varphi \big(K_\sigma^{\varphi,-}\big) \otimes \big(K_\sigma^{\varphi,+}\big) \widehat{E}_\sigma^\varphi\Big) \otimes \Big({\widehat{F}_\sigma^\varphi}{}' \big(K_\sigma^{\varphi,-\prime}\big) \otimes \big(K_\sigma^{\varphi,+\prime}\big) {\widehat{E}_\sigma^\varphi}{}'\Big) \tag{5.14}$$

where $a_\sigma^- \in k[q,q^{-1}]$ and the occurring PBW monomials have degree $-f\alpha_i$. Now comparison between (5.13) and (5.14) gives again a relation $a_\sigma^- = \big(q-q^{-1}\big)^{\mathrm{e}_\sigma} \cdot b_\sigma \cdot c_\sigma^-$ where $b_\sigma \in k[q,q^{-1}]$ and $\mathrm{e}_\sigma := \sum_{s=1}^N \big(f_s + e_s + f'_s + e'_s\big)$, with $\widehat{F}_\sigma^\varphi = \prod_{h=N}^1 \big(F_{\alpha^h}^\varphi\big)^{(f_h)}$, $\widehat{E}_\sigma^\varphi = \prod_{k=1}^N \big(E_{\alpha^k}^\varphi\big)^{(e_k)}$ and $\big(\widehat{F}_\sigma^\varphi\big)' = \prod_{h=N}^1 \big(F_{\alpha^h}^\varphi\big)^{(f'_h)}$, $\big(\widehat{E}_\sigma^\varphi\big)' = \prod_{k=1}^N \big(E_{\alpha^k}^\varphi\big)^{(e'_k)}$. Therefore

$$\Delta\Big(\big(F_i^\varphi\big)^{(f)} \otimes 1\Big) = \sum_{\kappa=0}^{\infty} \big(q-q^{-1}\big)^\kappa d_\kappa \cdot \Big(\widehat{F}_\kappa^\varphi \big(K_\kappa^{\varphi,-}\big) \otimes \big(K_\kappa^{\varphi,+}\big) \widehat{E}_\kappa^\varphi\Big) \otimes \Big({\widehat{F}_\kappa^\varphi}{}' \big(K_\kappa^{\varphi,-\prime}\big) \otimes \big(K_\kappa^{\varphi,+\prime}\big) {\widehat{E}_\kappa^\varphi}{}'\Big)$$

($d_\kappa \in k[q,q^{-1}]$, $\forall\, \kappa \in \mathbb{N}$), a convergent series in the $(q-1)$-adic topology of $\widetilde{A}_\infty^{\varphi,Q} \,\widehat{\otimes}\, \widetilde{A}_\infty^{\varphi,Q}$. Similarly, we find that $\Delta\big(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\big)$, $\Delta\big(K_{-(1+\varphi)(-\alpha_i)} \otimes K_{(1-\varphi)(-\alpha_i)}\big)$, $\Delta\Big(\Big({K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)};c \atop t}\Big)\Big)$ and $\Delta\Big(1 \otimes \big(E_i^\varphi\big)^{(e)}\Big)$ are expressed by convergent series in the $(q-1)$-adic topology of $\widetilde{A}_\infty^{\varphi,Q} \,\widehat{\otimes}\, \widetilde{A}_\infty^{\varphi,Q}$. In particular, all such series are in fact *finite* sums modulo $(q-1)$, whose first terms are given (proceeding like for S) by

$$\begin{aligned}
&\Delta\big(F_i^\varphi \otimes 1\big) \equiv \big(F_i^\varphi \otimes 1\big) \otimes (1 \otimes 1) + \\
&\quad + \big(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\big) \otimes \big(F_i^\varphi \otimes 1\big) && \mod (q-1)^2 \\
&\Delta\big(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\big) \equiv \\
&\quad \equiv \big(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\big) \otimes \big(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\big) && \mod (q-1)^2 \\
&\Delta\big(K_{-(1+\varphi)(-\alpha_i)} \otimes K_{(1-\varphi)(-\alpha_i)}\big) \equiv \\
&\quad \equiv \big(K_{-(1+\varphi)(-\alpha_i)} \otimes K_{(1-\varphi)(-\alpha_i)}\big) \otimes \big(K_{-(1+\varphi)(-\alpha_i)} \otimes K_{(1-\varphi)(-\alpha_i)}\big) && \mod (q-1)^2 \\
&\Delta\left(\left({K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)};0 \atop 1}\right)\right) \equiv \\
&\quad \equiv \left({K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)};0 \atop 1}\right) \otimes \big(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\big) + \\
&\qquad + (1 \otimes 1) \otimes \left({K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)};0 \atop 1}\right) && \mod (q-1)
\end{aligned}$$

$$\begin{aligned}
&\Delta\big(1 \otimes E_i^\varphi\big) \equiv (1 \otimes 1) \otimes \big(1 \otimes E_i^\varphi\big) + \\
&\quad + \big(1 \otimes E_i^\varphi\big) \otimes \big(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\big) && \mod (q-1)^2
\end{aligned}$$

(for all $i = 1, \dots, n$); moreover, for later use we also record that the same analysis gives

$$\begin{aligned}
&\Delta\big(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\big) \equiv \big(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\big) \otimes \big(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\big) + \\
&\quad + \sum_{s=1}^N \big(q_{\alpha^s} - q_{\alpha^s}^{-1}\big)^2 \cdot \big[\langle \alpha_i, \alpha^s \rangle\big]_{q_{\alpha^s}} \cdot \big(K_{-(1+\varphi)(\alpha^s)} \otimes K_{(1-\varphi)(\alpha^s)} E_{\alpha^s}^\varphi\big) \otimes \\
&\qquad \otimes \big(F_{\alpha^s}^\varphi K_{-(1+\varphi)(\alpha^s)} \otimes K_{(1-\varphi)(\alpha^s)}\big) && \mod (q-1)^4
\end{aligned}$$

from which we immediately get

$$\Delta\left(\begin{pmatrix} K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)};0 \\ 1 \end{pmatrix}\right) =$$
$$= \begin{pmatrix} K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)};0 \\ 1 \end{pmatrix} \otimes \left(K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)}\right) +$$
$$+ (1 \otimes 1) \otimes \begin{pmatrix} K_{-(1+\varphi)(\alpha_i)} \otimes K_{(1-\varphi)(\alpha_i)};0 \\ 1 \end{pmatrix} +$$
$$+ \sum_{s=1}^{N} (d_{\alpha^s})_q \, (q_{\alpha^s} - 1)(1 + q_{\alpha^s}^{-1})^2 \cdot [\langle \alpha_i, \alpha^s \rangle]_{q_{\alpha^s}} \cdot \big(K_{-(1+\varphi)(\alpha^s)} \otimes$$
$$\otimes K_{(1-\varphi)(\alpha^s)} E^{\varphi}_{\alpha^s}\big) \otimes \big(F^{\varphi}_{\alpha^s} K_{-(1+\varphi)(\alpha^s)} \otimes K_{(1-\varphi)(\alpha^s)}\big) \qquad \mod (q-1)^3 \, .$$

Remark: in the Appendix we shall explicitly write down the formulas for S and Δ in the case of $SL(2,k)$. In any case the technique we outlined here can be succesfully followed in order to compute the higher order terms of the series above up to any fixed degree.

The computations above have an important consequence; first a definition:

Definition 5.16. *We call $\widetilde{A}^{\varphi,Q}_{(q-1),\infty}$ the $(q-1)$-adic completion (which is naturally embedded into $\widetilde{A}^{\varphi,Q}_{\infty}$) of the $k[q,q^{-1}]$-algebra $\widetilde{A}^{\varphi,Q}$ defined in §11.9.* $\square$

The previous results about the (formal) Hopf structure of $\widetilde{A}^{\varphi,Q}_{\infty}$ then imply

$$\widetilde{A}^{\varphi,Q}_{(q-1),\infty} \;\; \textit{is a formal Hopf subalgebra of} \;\; \widetilde{A}^{\varphi,Q}_{\infty} \;\; \textit{over} \;\; k[q,q^{-1}] \, . \tag{5.15}$$

Moreover, we can improve Theorem 5.13 *(b2)* by the following

Theorem 5.17. *The algebra $\widetilde{F}^{Q}_{q,\varphi}[G]\left[\psi_{-\delta}^{-1}\right]$ is naturally embedded into $\widetilde{F}^{Q}_{q,\varphi}[G]_{(q-1),\infty}$; it follows that the $k[q,q^{-1}]$-algebra isomorphism $\mu^{\varphi,Q} \colon \widetilde{F}^{Q}_{q,\varphi}[G]\left[\psi_{-\delta}^{-1}\right] \xrightarrow{\cong} \widetilde{A}^{\varphi,Q}$ (cf. Theorem 5.13 (b1)) continuosly extends to an isomorphism of topological Hopf algebras over $k[q,q^{-1}]$*

$$\mu^{\varphi,Q}_{(q-1),\infty} \colon \widetilde{F}^{Q}_{q,\varphi}[G]_{(q-1),\infty} \xrightarrow{\cong} \widetilde{A}^{\varphi,Q}_{(q-1),\infty} \, .$$

Proof. Let consider the case $\varphi = 0$. From the proof of Theorem 5.13 we recall that
$$\mu^{Q}(\psi_{-\delta}) = K_{-\delta} \otimes K_{\delta} = \textstyle\prod_{i=1}^{n} K_i^{-1} \otimes K_i = \prod_{i=1}^{n} \mu^{Q}(\psi_{-\alpha_i})$$
(cf. (5.9)); since definitions give
$$K_i^{-1} \otimes K_i = 1 + (q_i - 1)\begin{pmatrix} K_i^{-1} \otimes K_i;0 \\ 1 \end{pmatrix} = 1 - (-1)(q_i - 1)\begin{pmatrix} K_i^{-1} \otimes K_i;0 \\ 1 \end{pmatrix}$$
we have

$$\begin{aligned} \left(K_i^{-1} \otimes K_i\right)^{-1} &= \left(1 - (-1)(q_i - 1)\begin{pmatrix} K_i^{-1} \otimes K_i;0 \\ 1 \end{pmatrix}\right)^{-1} = \\ &= \sum_{n=0}^{\infty} (-1)^n (q_i - 1)^n \cdot \begin{pmatrix} K_i^{-1} \otimes K_i;0 \\ 1 \end{pmatrix}^n \in \widetilde{A}^{Q}_{(q-1),\infty} \end{aligned} \tag{5.16}$$

but extending μ^Q to $\widetilde{F}_q^Q[G]_{(q-1),\infty}$ one gets

$$\mu^Q_{(q-1),\infty}\left(\sum_{n=0}^{\infty}(-1)^n(q_i-1)^n\cdot\binom{z_i;0}{1}^n\right)=\sum_{n=0}^{\infty}(-1)^n(q_i-1)^n\cdot\binom{K_i^{-1}\otimes K_i;0}{1}^n$$

this along with (5.16) and injectivity of $\mu^Q_\infty\colon \widetilde{F}_q^{Q,\infty}[G]\stackrel{\cong}{\longrightarrow}\widetilde{A}^Q_\infty$ (cf. Theorem 5.13) gives

$$\psi_{-\delta}^{-1}=\sum_{n=0}^{\infty}(-1)^n(q_i-1)^n\cdot\binom{z_i;0}{1}^n\in\widetilde{F}_q^Q[G]_{(q-1),\infty}\,.$$

The same argument also works in the general case, replacing $K_i^{-1}\otimes K_i$ with $K_{-(1+\varphi)(\alpha_i)}\otimes K_{(1-\varphi)(\alpha_i)}$. The thesis follows. $\square$

§ 6 Quantization of $U(\mathfrak{h}^\tau)$

6.1 The quantum algebra $U_{q,\varphi}^\infty(\mathfrak{h})$. The results we discussed in §5 can acquire more brightness by means of an axiomatic approach: to this end, we introduce now a new object $U_{q,\varphi}^{M,\infty}(\mathfrak{h})$, by means of a presentation by generators and relations, which is with respect to $U(\mathfrak{h}^\tau)$ what $U_{q,\varphi}^M(\mathfrak{g})$ is with respect to $U(\mathfrak{g}^\tau)$.

Fix a lattice M ($Q\leq M\leq P$); let $U_{q,\varphi}^M(\mathfrak{h})_0$ be the associative $k(q)$-algebra with 1 with generators

$$F_i^\varphi\,,\ L_\mu^\varphi\,,\ E_i^\varphi\quad(\mu\in M;i=1,\dots,n)$$

and relations

$$\begin{gathered}
L_0^\varphi=1\,,\qquad L_\mu^\varphi L_\nu^\varphi=L_{\mu+\nu}^\varphi\\
L_\mu^\varphi F_j^\varphi=q^{(\alpha_j|(1+\varphi)(\mu))}F_j^\varphi L_\mu^\varphi\,,\qquad L_\mu^\varphi E_j^\varphi=q^{(\alpha_j|(1-\varphi)(\mu))}E_j^\varphi L_\mu^\varphi\\
E_i^\varphi F_h^\varphi-F_h^\varphi E_i^\varphi=0\\
\sum_{k=0}^{1-a_{ij}}(-1)^k q^{+c_{ij}^k}\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}(E_i^\varphi)^{1-a_{ij}-k}E_j^\varphi(E_i^\varphi)^k=0\\
\sum_{k=0}^{1-a_{ij}}(-1)^k q^{-c_{ij}^k}\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}(F_i^\varphi)^{1-a_{ij}-k}F_j^\varphi(F_i^\varphi)^k=0
\end{gathered}\tag{6.1}$$

where $c_{ij}^k:=-(k\alpha_i|\tau_j+(1-a_{ij}-k)\tau_i)-(\alpha_j|(1-a_{ij}-k)\tau_i)$ and $i,j,h=1,\dots,n\,,\,i\neq j$, $\mu,\nu\in M$. Then let $\varepsilon'\colon U_{q,\varphi}^M(\mathfrak{h})_0\to k(q)$ be the $k(q)$-algebra morphism defined by

$$\varepsilon'\left(F_i^\varphi\right):=0\,,\ \varepsilon'\left(L_\mu^\varphi\right):=1\,,\ \varepsilon'\left(E_i^\varphi\right):=0\,,\quad\forall\,i=1,\dots,n,\,\mu\in M$$

and set $\mathfrak{E}:=Ker(\varepsilon')$. Finally, we define $U_{q,\varphi}^{M,\infty}(\mathfrak{h})$ to be the $\mathfrak{E}$-adic completion of $U_{q,\varphi}^M(\mathfrak{h})_0$, with its natural structure of topological $k(q)$-algebra. Roughly speaking, this is an algebra of (non-commutative) formal series in the given generators with the (6.1) as commutation

rules; in short, $U_{q,\varphi}^{M,\infty}(\mathfrak{h})$ is the topological algebra over $k(q)$ generated by F_i^φ, L_μ^φ, E_i^φ $(\mu \in M; i = 1, \dots, n)$ with relations (6.1). When $M = Q$ we shall also set $K_\alpha^\varphi := L_\alpha^\varphi$ for all $\alpha \in Q$ (and more in general $L_i^\varphi := L_{\omega_i}^\varphi$, $K_i^\varphi := K_{\alpha_i}^\varphi$, for all $i = 1, \dots, n$).

The key point now is that results in §5 — especially Proposition 5.4 — provides a concrete realization of the algebras $U_{q,\varphi}^{M,\infty}(\mathfrak{h})$, while endowing them with a structure of formal Hopf algebra too:

Theorem 6.2. *The rule*

$$\nu_M^\varphi\colon \quad F_i^\varphi \mapsto F_i^\varphi \otimes 1\,, \quad L_\lambda^\varphi \mapsto L_{-(1+\varphi)(\lambda)}^\varphi \otimes L_{(1-\varphi)(\lambda)}^\varphi\,, \quad E_i^\varphi \mapsto 1 \otimes E_i^\varphi$$

$(\forall\, i = 1, \dots, n,\ \lambda \in M)$ defines a $k(q)$-algebra isomorphism $\nu_M^\varphi \colon U_{q,\varphi}^{M,\infty}(\mathfrak{h}) \xrightarrow{\cong} A_\infty^{\varphi,M}$; then the pull-back of the Hopf structure of $A_\infty^{\varphi,M}$ defines a formal Hopf algebra structure on $U_{q,\varphi}^{M,\infty}(\mathfrak{h})$, so that ν_M^φ is an isomorphism of formal Hopf algebras over $k(q)$.

Proof. Trivial from the very definitions. $\square$

6.3 Integer forms of $U_{q,\varphi}^{M,\infty}(\mathfrak{h})$. In this subsection we introduce integer forms of $U_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ and $U_{q,\varphi}^{P,\infty}(\mathfrak{h})$; in this formal approach, it can be done like for $U_{q,\varphi}^{M}(\mathfrak{g})$ in §3.4.

Let $\widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ be the complete $k[q,q^{-1}]$-subalgebra of $U_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ generated by

$$\left\{ (F_i^\varphi)^{(\ell)}, \begin{pmatrix} K_i^\varphi ; c \\ t \end{pmatrix}, (E_i^\varphi)^{(m)} \,\middle|\, \ell, c, t, m \in \mathbb{N}; i = 1, \dots, n \right\} \tag{6.2}$$

where $(F_i^\varphi)^{(\ell)}$, $\begin{pmatrix} K_i^\varphi ; c \\ t \end{pmatrix}$, and $(E_i^\varphi)^{(m)}$ are usual divided powers (we do not need the $(K_i^\varphi)^{-1}$'s among generators, because $(K_i^\varphi)^{-1} = (1-(1-K_i^\varphi))^{-1} = \sum_{n=0}^\infty (1-K_i^\varphi)^n$); one can easily check — for instance looking at quantum Borel subalgebras, which are naturally embedded inside $\widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ as well as inside $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{g})$, and exploiting the analogous results for $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{g})$ — that $\widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ has a PBW pseudobasis (over $k[q,q^{-1}]$)

$$\left\{ \prod_{r=N}^{1} (F_{\alpha^r}^\varphi)^{(m_r)} \cdot \prod_{i=1}^{n} \begin{pmatrix} K_i^\varphi ; 0 \\ t_i \end{pmatrix} \cdot \prod_{r=1}^{N} (E_{\alpha^r}^\varphi)^{(n_r)} \,\middle|\, m_r, t_i, n_r \in \mathbb{N}\ \forall\, r, \forall\, i \right\} . \tag{6.3}$$

Then $\widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ is an integer form (over $k[q,q^{-1}]$) of $U_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ *in topological sense*, that is $k(q) \otimes_{k[q,q^{-1}]} \widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ is dense in $U_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ (with respect to the $\mathfrak{E}$-adic topology).

Now let $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h})$ be the complete $k[q,q^{-1}]$-subalgebra of $U_{q,\varphi}^{P,\infty}(\mathfrak{h})$ generated by

$$\left\{ \overline{F}_{\alpha^1}^\varphi, \dots, \overline{F}_{\alpha^N}^\varphi \right\} \cup \left\{ (L_1^\varphi)^{\pm 1}, \dots, (L_n^\varphi)^{\pm 1} \right\} \cup \left\{ \overline{F}_{\alpha^1}^\varphi, \dots, \overline{E}_{\alpha^N}^\varphi \right\}$$

where $\overline{F}_{\alpha^r}^\varphi := (q_{\alpha^r} - q_{\alpha^r}^{-1}) F_{\alpha^r}^\varphi$, $\overline{E}_{\alpha^r}^\varphi := (q_{\alpha^r} - q_{\alpha^r}^{-1}) E_{\alpha^r}^\varphi$, $\forall\, r = 1, \dots, N$, as usual; one checks — again via Borel subalgebras — that $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h})$ has a PBW pseudobasis (over $k[q,q^{-1}]$)

$$\left\{ \prod_{r=N}^{1} \left(\overline{F}_{\alpha^r}^\varphi\right)^{m_r} \cdot \prod_{i=1}^{n} (L_i^\varphi)^{t_i} \cdot \prod_{s=1}^{N} \left(\overline{E}_{\alpha^s}^\varphi\right)^{n_s} \,\middle|\, m_r, t_i, n_s \in \mathbb{N}\ \forall\, r, i, s \right\}$$

therefore $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h})$ is a $k[q,q^{-1}]$-integer form of $U_{q,\varphi}^{P,\infty}(\mathfrak{h})$ (in topological sense).

Finally, we define $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})_{(q-1),\infty}$ to be the $(q-1)$-adic completion of the $k[q,q^{-1}]$-subalgebra of $U_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ generated by the set in (6.2); then $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})_{(q-1),\infty}$ is a $k[q,q^{-1}]$-subalgebra of $\widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h})$, with (6.3) as a PBW pseudobasis (as a topological $k[q,q^{-1}]$-module); notice also that $(K_i^\varphi)^{-1} \in \widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h})$, for $(K_i^\varphi)^{-1} = (1-(1-K_i^\varphi))^{-1} = \sum_{n=0}^{\infty}(1-K_i^\varphi)^n = \sum_{n=0}^{\infty}(-1)(q_i-1)\binom{K_i^\varphi;0}{1}^n \in \widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})_{(q-1),\infty}$. In particular $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})_{(q-1),\infty}$ is an integer form of $U_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ (in topological sense). In the sequel we shall also use the notation $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h}) := \widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})_{(q-1),\infty}$.

Now from the very definitions, Proposition 5.9, Definition 5.16, and (5.15), we easily get the following (notice the reversal of accents " $\widehat{\ }$ " and " $\widetilde{\ }$ "):

Proposition 6.4.

The formal Hopf algebra isomorphism $\nu_Q^\varphi \colon U_{q,\varphi}^{Q,\infty}(\mathfrak{h}) \overset{\cong}{\to} A_\infty^{\varphi,Q}$, *resp.* $\nu_P^\varphi \colon U_{q,\varphi}^{P,\infty}(\mathfrak{h}) \overset{\cong}{\to} A_\infty^{\varphi,P}$, *resp.* $\nu_Q^\varphi \colon U_{q,\varphi}^{Q,\infty}(\mathfrak{h}) \overset{\cong}{\to} A_\infty^{\varphi,Q}$, *maps* $\widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h})$, *resp.* $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h})$, *resp.* $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})_{(q-1),\infty}$, *isomorphically onto* $\widetilde{A}_\infty^{\varphi,Q}$, *resp.* $\widehat{A}_\infty^{\varphi,P}$, *resp.* $\widetilde{A}_{(q-1),\infty}^{\varphi,Q}$; *in particular,* $\widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h})$, *resp.* $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h})$, *resp.* $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})_{(q-1),\infty}$, *is a formal Hopf subalgebra, hence a (topological)* $k[q,q^{-1}]$*-form of the formal Hopf algebra* $U_{q,\varphi}^{Q,\infty}(\mathfrak{h})$, *resp.* $U_{q,\varphi}^{P,\infty}(\mathfrak{h})$, *resp.* $U_{q,\varphi}^{Q,\infty}(\mathfrak{h})$. $\square$

6.5. Presentation of $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})$. Exploiting the analogous result for $\widehat{U}_q^Q(\mathfrak{g})$ (cf. [DL], §3.4), we can present $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h}) := \widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})_{(q-1),\infty}$ as the $(q-1)$-adic completion of the associative $k[q,q^{-1}]$-algebra with 1 generated by K_i^φ, $(K_i^\varphi)^{-1}$, $\binom{K_i^\varphi;c}{t}$, $(E_i^\varphi)^{(r)}$, $(F_i^\varphi)^{(s)}$ ($c\in\mathbb{Z}$, $t,r,s\in\mathbb{N}$, $i=1,\dots,n$) with relations

$$K_i^\varphi (K_i^\varphi)^{-1} = 1 = (K_i^\varphi)^{-1} K_i^\varphi \, , \; (K_i^\varphi)^{\pm 1} (K_j^\varphi)^{\pm 1} = (K_j^\varphi)^{\pm 1} (K_i^\varphi)^{\pm 1} \, , \qquad \forall\, i,j$$

$$(K_i^\varphi)^{\pm 1} \binom{K_j^\varphi;c}{t} = \binom{K_j^\varphi;c}{t} (K_i^\varphi)^{\pm 1} \, , \qquad \forall\, i,j$$

$$\binom{K_i^\varphi;c}{0} = 0 \, , \; (q_i-1)\binom{K_i^\varphi;0}{1} = K_i^\varphi - 1 \, , \qquad \forall\, i$$

$$\binom{K_i^\varphi;c}{t}\binom{K_i^\varphi;c-t}{s} = \binom{t+s}{t}_{q_i}\binom{K_i^\varphi;c}{t+s} \, , \qquad \forall ; , i,t,s$$

$$\binom{K_i^\varphi;c+1}{t} - q_i^t \binom{K_i^\varphi;c}{t} = \binom{K_i^\varphi;c}{t-1} \, , \qquad \forall\, t \geq 1$$

$$\binom{K_i^\varphi;c}{t} = \sum_{p\geq 0}^{p\leq c,t} q_i^{(c-p)(t-p)} \binom{c}{p}_{q_i} \binom{K_i^\varphi;0}{t-1} \, , \qquad \forall\, c \geq 0$$

$$\binom{K_i^\varphi ; -c}{t} = \sum_{p=0}^{t} (-1)^p q_i^{-t(c+p)+p(p+1)/2} \binom{p+c-1}{p}_{q_i} \binom{K_i^\varphi ; 0}{t-p}, \qquad \forall\, c \geq 1$$

$$\binom{K_i^\varphi ; c+1}{t} - \binom{K_i^\varphi ; c}{t} = q_i^{c-t+1} K_i^\varphi \binom{K_i^\varphi ; c}{t-1}, \qquad \forall\, t \geq 1$$

$$K_i^\varphi (E_j^\varphi)^{(p)} = q^{p(\alpha_j|\alpha_i - 2\tau_i)} (E_j^\varphi)^{(p)}, \qquad \forall\, i,j$$

$$K_i^\varphi (F_j^\varphi)^{(p)} = q^{p(\alpha_j|\alpha_i + 2\tau_i)} (F_j^\varphi)^{(p)}, \qquad \forall\, i,j$$

$$\binom{K_i^\varphi ; c}{t} (E_j^\varphi)^{(p)} = (E_j^\varphi)^{(p)} \binom{K_i^\varphi ; c + p d_i^{-1} (\alpha_j|\alpha_i - 2\tau_i)}{t}, \qquad \forall\, i,j$$

$$\binom{K_i^\varphi ; c}{t} (F_j^\varphi)^{(p)} = (F_j^\varphi)^{(p)} \binom{K_i^\varphi ; c + p d_i^{-1} (\alpha_j|\alpha_i + 2\tau_i)}{t}, \qquad \forall\, i,j$$

$$(E_i^\varphi)^{(r)} (E_i^\varphi)^{(s)} = \begin{bmatrix} r+s \\ r \end{bmatrix}_{q_i} (E_i^\varphi)^{(r+s)}, \quad (E_i^\varphi)^{(0)} = 1 \qquad \forall\, i$$

$$(F_i^\varphi)^{(r)} (F_i^\varphi)^{(s)} = \begin{bmatrix} r+s \\ r \end{bmatrix}_{q_i} (F_i^\varphi)^{(r+s)}, \quad (F_i^\varphi)^{(0)} = 1 \qquad \forall\, i$$

$$\sum_{r+s=1-a_{ij}} (-1)^s q^{-(k\alpha_i|\tau_j + (1-a_{ij}-k)\tau_i) - (\alpha_j|(1-a_{ij}-k)\tau_i)} (E_i^\varphi)^{(r)} E_j^\varphi (E_i^\varphi)^{(s)} = 0, \quad \forall\, i \neq j$$

$$\sum_{r+s=1-a_{ij}} (-1)^s q^{(k\alpha_i|\tau_j + (1-a_{ij}-k)\tau_i) + (\alpha_j|(1-a_{ij}-k)\tau_i)} (F_i^\varphi)^{(r)} F_j^\varphi (F_i^\varphi)^{(s)} = 0, \quad \forall\, i \neq j$$

$$(E_i^\varphi)^{(r)} (F_j^\varphi)^{(s)} = (F_j^\varphi)^{(s)} (E_i^\varphi)^{(r)}, \qquad \forall\, i,j;$$

in fact it is enough to use generators $\binom{K_i^\varphi;0}{t}$, $(E_i^\varphi)^{(r)}$, $(F_i^\varphi)^{(s)}$. Then computations in §5.13 — via the isomorphism ν_Q^φ — yield the following (for all $i = 1, \dots, n$):

$$\Delta(F_i^\varphi) \equiv F_i^\varphi \otimes 1 + K_i^\varphi \otimes F_i^\varphi \qquad \mod (q-1)^2$$

$$\Delta(K_i^\varphi) \equiv K_i^\varphi \otimes K_i^\varphi \qquad \mod (q-1)^2$$

$$\Delta\left((K_i^\varphi)^{-1}\right) \equiv (K_i^\varphi)^{-1} \otimes (K_i^\varphi)^{-1} \qquad \mod (q-1)^2$$

$$\Delta\left(\binom{K_i^\varphi;0}{1}\right) \equiv \binom{K_i^\varphi;0}{1} \otimes K_i^\varphi + 1 \otimes \binom{K_i^\varphi;0}{1} +$$
$$+ \sum_{s=1}^{N} (d_{\alpha^s})_q (q_{\alpha^s} - 1)(1 + q_{\alpha^s}^{-1})^2 \cdot [\langle \alpha_i, \alpha^s \rangle]_{q_{\alpha^s}} \cdot K_{\alpha^s}^\varphi E_{\alpha^s}^\varphi \otimes F_{\alpha^s}^\varphi K_{\alpha^s}^\varphi \qquad \mod (q-1)^3$$

$$\Delta(E_i^\varphi) \equiv E_i^\varphi \otimes K_i^\varphi + 1 \otimes E_i^\varphi \qquad \mod (q-1)^2$$

$$\varepsilon(F_i^\varphi) = 0, \qquad \varepsilon(E_i^\varphi) = 0$$

$$\varepsilon(K_i^\varphi) = 1, \qquad \varepsilon\left(\binom{K_i^\varphi;0}{1}\right) = 0, \qquad \varepsilon\left((K_i^\varphi)^{-1}\right) = 1$$

$$\begin{aligned}
S\left(F_i^\varphi\right) &\equiv -q_i^{-2} \cdot F_i^\varphi \left(K_i^\varphi\right)^{-1} && \mod (q-1)^2 \\
S\left(K_i^\varphi\right) &\equiv \left(K_i^\varphi\right)^{-1} && \mod (q-1)^2 \\
S\left(\left(K_i^\varphi\right)^{-1}\right) &\equiv K_i^\varphi && \mod (q-1)^2 \\
S\left(E_i^\varphi\right) &\equiv -q_i^{+2} \cdot \left(K_i^\varphi\right)^{-1} E_i^\varphi && \mod (q-1)^2
\end{aligned}$$

§ 7 Quantum Poisson pairing

7.1 The natural embedding $\xi_\varphi^M \colon F_{q,\varphi}^M[G] \longrightarrow U_{q,\varphi}^\infty(\mathfrak{h})$. In this section we fit together the results of §5 and §6 to construct perfect Hopf pairings $U_{q,\varphi}^{M,\infty}(\mathfrak{h}) \otimes U_{q,\varphi}^{M'}(\mathfrak{g}) \rightarrow k(q)$ which — in a sense to be explained in §8 — will provide a quantization of three classical objects: the Hopf pairings $F[G^\tau] \otimes U(\mathfrak{g}^\tau) \rightarrow k$ (or $F^\infty[G^\tau] \otimes U(\mathfrak{g}^\tau) \rightarrow k$) and $U(\mathfrak{h}^\tau) \otimes F[H^\tau] \rightarrow k$ (or $U(\mathfrak{h}^\tau) \otimes F^\infty[H^\tau] \rightarrow k$) and the Lie bialgebra pairing $\mathfrak{h}^\tau \otimes \mathfrak{g}^\tau \rightarrow k$; therefore we shall call them "quantum Poisson pairing". We define formal Hopf algebra embeddings

$$\xi_\varphi^M := \left(\nu_M^\varphi\right)^{-1} \circ \mu_\varphi^{M'} \colon F_{q,\varphi}^{M'}[G] \hookrightarrow U_{q,\varphi}^{M,\infty}(\mathfrak{h})$$

and review results of §5 in terms of §6; in particular, we single out the following:

Theorem 7.2.
(a) The embedding $\xi_\varphi^P \colon F_{q,\varphi}^P[G] \hookrightarrow U_{q,\varphi}^{P,\infty}(\mathfrak{h})$ induces embeddings of algebras

$$\xi_\varphi^P \colon F_{q,\varphi}^P[G] \hookrightarrow U_{q,\varphi}^P(\mathfrak{h})_0\,, \qquad \xi_\varphi^P \colon \widehat{F}_{q,\varphi}^P[G] \hookrightarrow \widetilde{U}_{q,\varphi}^P(\mathfrak{h})_0$$

isomorphisms of algebras

$$\xi_\varphi^P \colon F_{q,\varphi}^P[G]\left[\psi_{-\rho}^{-1}\right] \xrightarrow{\cong} U_{q,\varphi}^P(\mathfrak{h})_0\,, \qquad \xi_\varphi^P \colon \widehat{F}_{q,\varphi}^P[G]\left[\psi_{-\rho}^{-1}\right] \xrightarrow{\cong} \widetilde{U}_{q,\varphi}^P(\mathfrak{h})_0$$

and isomorphisms of formal Hopf algebras

$$\xi_\varphi^P \colon F_{q,\varphi}^{P,\infty}[G] \xrightarrow{\cong} U_{q,\varphi}^{P,\infty}(\mathfrak{h})\,, \qquad \xi_\varphi^P \colon \widehat{F}_{q,\varphi}^{P,\infty}[G] \xrightarrow{\cong} \widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h})\,;$$

in particular $\widehat{F}_{q,\varphi}^{P,\infty}[G]$ is an integer form of $F_{q,\varphi}^{P,\infty}[G]$ (in topological sense).
(b) The embedding $\xi_\varphi^Q \colon F_{q,\varphi}^Q[G] \hookrightarrow U_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ induces embeddings of algebras

$$\xi_\varphi^Q \colon F_{q,\varphi}^Q[G] \hookrightarrow U_{q,\varphi}^Q(\mathfrak{h})_0\,, \qquad \xi_\varphi^Q \colon \widetilde{F}_{q,\varphi}^Q[G] \hookrightarrow \widehat{U}_{q,\varphi}^Q(\mathfrak{h})_0$$

isomorphisms of algebras

$$\xi_\varphi^Q \colon F_{q,\varphi}^Q[G]\left[\psi_{-\delta}^{-1}\right] \xrightarrow{\cong} U_{q,\varphi}^Q(\mathfrak{h})_0\,, \qquad \xi_\varphi^Q \colon \widetilde{F}_{q,\varphi}^Q[G][\psi_{-\delta}^{-1}] \xrightarrow{\cong} \widehat{U}_{q,\varphi}^Q(\mathfrak{h})_0$$

and isomorphisms of formal Hopf algebras

$$\xi_\varphi^Q \colon F_{q,\varphi}^{Q,\infty}[G] \xrightarrow{\cong} U_{q,\varphi}^{Q,\infty}(\mathfrak{h})\,, \qquad \xi_\varphi^Q \colon \widetilde{F}_{q,\varphi}^{Q,\infty}[G] \xrightarrow{\cong} \widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h})$$

$$\xi_\varphi^Q \colon F_{q,\varphi}^Q[G]_{(q-1),\infty} \xrightarrow{\cong} U_{q,\varphi}^Q(\mathfrak{h})_{(q-1),\infty}\,, \qquad \xi_\varphi^Q \colon \widetilde{F}_{q,\varphi}^Q[G]_{(q-1),\infty} \xrightarrow{\cong} \widehat{U}_{q,\varphi}^Q(\mathfrak{h})_{(q-1),\infty}$$

where $F_{q,\varphi}^Q[G]_{(q-1),\infty} := k(q) \otimes_{k[q,q^{-1}]} \widetilde{F}_{q,\varphi}^Q[G]_{(q-1),\infty}$ and $U_{q,\varphi}^Q(\mathfrak{h})_{(q-1),\infty} := k(q) \otimes_{k[q,q^{-1}]} \widehat{U}_{q,\varphi}^Q(\mathfrak{h})_{(q-1),\infty}$; in particular $\widetilde{F}_{q,\varphi}^{Q,\infty}[G]$ and $\widetilde{F}_{q,\varphi}^Q[G]_{(q-1),\infty}$ are integer forms of $F_{q,\varphi}^{Q,\infty}[G]$ (in topological sense). $\square$

7.3 The quantum Poisson pairing. Since $A_\infty^{\varphi,M}$ is a formal Hopf subalgebra of $U_{q,\varphi}^{M'}(\mathfrak{g})^*$ (cf. §5), restriction of the evaluation pairing yields perfect Hopf pairings

$$\langle\ ,\ \rangle \colon A_\infty^{\varphi,P} \otimes U_{q,\varphi}^Q(\mathfrak{g}) \rightarrow k(q)\,, \qquad \langle\ ,\ \rangle \colon A_\infty^{\varphi,Q} \otimes U_{q,\varphi}^P(\mathfrak{g}) \rightarrow k(q)\,.$$

Exploiting the isomorphisms ν_M^φ we then fix the following

Definition 7.4. *We define* **quantum Poisson pairing** *the perfect pairing of formal Hopf algebras* $\pi_q^\varphi \colon U_{q,\varphi}^{M,\infty}(\mathfrak{h}) \otimes U_{q,\varphi}^{M'}(\mathfrak{g}) \to k(q)$ *given by* $\pi_q^\varphi(h,g) := \langle (\nu_M^\varphi)^{-1}(h), g \rangle$ *for all* $h \in U_{q,\varphi}^{M,\infty}(\mathfrak{h})$, $g \in U_{q,\varphi}^{M'}(\mathfrak{g})$. $\square$

In next paragraph we will show that this provides a quantization of the "classical" Poisson pairing $\pi_{\mathcal{P}}^\tau \colon \mathfrak{h}^\tau \otimes \mathfrak{g}^\tau \longrightarrow k$, whence our choice of the name; for the time being, we remark only that the previous analysis ensures that integer forms — $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h})$, $\widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h})$, $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{g})$ and $\widetilde{U}_{q,\varphi}^{P}(\mathfrak{g})$ — of our quantized enveloping algebras are the orthogonal of each other with respect to π_q^φ; furthermore, π_q^φ restrict to perfect Hopf pairings

$$\pi_{q,H^\tau}^\varphi \colon \widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h}) \otimes \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) \to k[q,q^{-1}] \, , \qquad \pi_{q,G^\tau}^\varphi \colon \widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h}) \otimes \widehat{U}_{q,\varphi}^{Q}(\mathfrak{g}) \to k[q,q^{-1}] \, ;$$

with the same symbol π_{q,G^τ}^φ we shall also denote the Hopf pairings $\pi_{q,G^\tau}^\varphi \colon \widehat{F}_{q,\varphi}^{P}[G] \otimes \widehat{U}_{q,\varphi}^{Q}(\mathfrak{g}) \longrightarrow k[q,q^{-1}]$, resp. $\pi_{q,H^\tau}^\varphi \colon \widetilde{F}_{q,\varphi}^{Q}[G] \otimes \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) \longrightarrow k[q,q^{-1}]$, gotten from restriction of $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h}) \otimes \widehat{U}_{q,\varphi}^{Q}(\mathfrak{g}) \to k[q,q^{-1}]$, resp. $\widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h}) \otimes \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) \to k[q,q^{-1}]$ (hereafter we identify $F_{q,\varphi}^M[G]$, with its image in $U_{q,\varphi}^{M,\infty}(\mathfrak{h})$ via ξ_φ^M, and similarly for integer forms); we shall also use the same notations when $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})$ replaces $\widehat{U}_{q,\varphi}^{Q,\infty}(\mathfrak{h})$.

§ 8 Specialization at roots of 1

8.1 The case $q \to 1$: specialization of $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})$ to $U(\mathfrak{h}^\tau)$. As in §3.5, we assume for simplicity that our ground field k contains all roots of unity (so that $k[q,q^{-1}] \big/ (q-\varepsilon) = k$ for any root of unity ε; otherwise, specializing q to ε will extend the ground field to $k[q,q^{-1}] \big/ p_\ell(q) = k(\varepsilon)$, where $p_\ell(q)$ is the ℓ-th cyclotomic polynomial); in particular this is the case for $k = \mathbb{C}$. Moreover, we fix $\tau = (\tau_1, \dots, \tau_n) := \frac{1}{2}(\varphi(\alpha_1), \dots, \varphi(\alpha_n))$.

To begin with, we consider specialization of $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h}) := \widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})_{(q-1),\infty}$ at $q=1$: set

$$\widehat{U}_{1,\varphi}^{Q}(\mathfrak{h}) := \widehat{U}_{q,\varphi}^{Q}(\mathfrak{h}) \Big/ (q-1)\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h}) \cong \widehat{U}_{q,\varphi}^{Q}(\mathfrak{h}) \otimes_{k[q,q^{-1}]} k$$

(k is a $k[q,q^{-1}]$-algebra via $k \cong k[q,q^{-1}] \big/ (q-1)$); let $p_1^\varphi \colon \widehat{U}_{q,\varphi}^{Q}(\mathfrak{h}) \twoheadrightarrow \widehat{U}_{1,\varphi}^{Q}(\mathfrak{h})$ be the canonical map and set $\mathrm{f}_i^\tau := p_1^\varphi\Big((F_i^\varphi)^{(1)} \Big)$, $\mathrm{h}_i^\tau := p_1^\varphi\Big(\Big({K_i^\varphi ; 0 \atop 1} \Big) \Big)$, $\mathrm{e}_i^\tau := p_1^\varphi\Big((E_i^\varphi)^{(1)} \Big)$ $(i = 1, \dots, n)$. Then one immediately finds that $\widehat{U}_{1,\varphi}^{Q}(\mathfrak{h})$ is cocommutative, hence inherits from $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})$ a canonical co-Poisson Hopf structure; furthermore, it also inherits from $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})$ a presentation by generators $(\mathrm{f}_i^\tau, \mathrm{h}_i^\tau, \mathrm{e}_i^\tau)$ and relations which is exactly the same as $U(\mathfrak{h}^\tau)$ given in (1.3–5). To give an example, we consider the case $\varphi = 0$ (hence $\tau = 0$)

and compute $\delta(\mathrm{h}_i)$; by definition of quantization of co-Poisson structure, δ is given by $\delta := \frac{\Delta - \Delta^{op}}{q-1}$ (where Δ^{op} denotes opposite comultiplication), whence we have

$$\delta(h_i) := \frac{\Delta\left(\binom{K_i;0}{1}\right) - \Delta^{op}\left(\binom{K_i;0}{1}\right)}{q-1}\Bigg|_{q=1} =$$

$$= \left(\sum_{s=1}^N (d_{\alpha^s})_q^2 \cdot \left(1 + q_{\alpha^s}^{-1}\right)^2 \cdot [\langle \alpha_i, \alpha^s \rangle]_{q_{\alpha^s}} \cdot \left(K_{\alpha^s} E_{\alpha^s} \otimes F_{\alpha^s} K_{\alpha^s} - F_{\alpha^s} K_{\alpha^s} \otimes K_{\alpha^s} E_{\alpha^s} \right) \right) \Bigg|_{q=1} =$$

$$= \sum_{s=1}^N d_{\alpha^s}^2 \cdot 2^2 \cdot \langle \alpha_i, \alpha^s \rangle \cdot \left(\mathrm{e}_{\alpha^s} \otimes \mathrm{f}_{\alpha^s} - \mathrm{f}_{\alpha^s} \otimes \mathrm{e}_{\alpha^s} \right) =$$

$$= 4 d_i^{-1} \cdot \sum_{s=1}^N d_{\alpha^s}^2 \cdot (\alpha^s | \alpha_i) \cdot \left(\mathrm{e}_{\alpha^s} \otimes \mathrm{f}_{\alpha^s} - \mathrm{f}_{\alpha^s} \otimes \mathrm{e}_{\alpha^s} \right) \qquad \text{q. e. d.}$$

Thus we have proved the following

Theorem 8.2. *The formal topological Hopf algebra $\widehat{U}_{q,\varphi}^{Q}(\mathfrak{h})$ specializes for $q \to 1$ to the Poisson Hopf coalgebra $U(\mathfrak{h}^\tau)$, i. e. there exists an isomorphism of Poisson Hopf coalgebras*

$$\widehat{U}_{1,\varphi}^{Q}(\mathfrak{h}) \cong U(\mathfrak{h}^\tau) \,. \, \square$$

By the way, this also makes it possible to recover without effort the dual result (3.8) $\widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) \xrightarrow{q \to 1} F[H^\tau]$ (cf. §3.5), whose original proof — more precisely, the proof of the part about Poisson structures — exhibited in [DKP] and [DP] is lenghty involved and complicated, requiring very hard computations; namely, we have the following corollary:

Corollary 8.3. *The Hopf algebra $\widetilde{U}_{q,\varphi}^{P}(\mathfrak{g})$ specializes to the Poisson Hopf algebra $F[H^\tau]$ for $q \to 1$; in other words there exists an isomorphism of Poisson Hopf algebras*

$$\widetilde{U}_{1,\varphi}^{P}(\mathfrak{g}) := \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) \otimes_{k[q,q^{-1}]} k \;\cong\; F[H^\tau] \,.$$

Proof. Since $\widetilde{U}_{1,\varphi}^{P}(\mathfrak{g})$ is perfectly paired with $\widehat{U}_{1,\varphi}^{Q}(\mathfrak{h}) \cong U(\mathfrak{h}^\tau)$ and the latter is cocommutative, the former is commutative; thus $\widetilde{U}_{1,\varphi}^{P}(\mathfrak{g})$ is a commutative Hopf algebra, finitely generated over k, hence it is the function algebra of an affine algebraic group, say $\mathcal{H}^\tau$; moreover, from its deformation $\widetilde{U}_{q,\varphi}^{P}(\mathfrak{g})$ the Hopf algebra $\widetilde{U}_{1,\varphi}^{P}(\mathfrak{g}) = F[\mathcal{H}^\tau]$ inherits — in the usual way — a Poisson structure, whence $\mathcal{H}^\tau$ is a Poisson group. As in [DP] the presentation of $\widetilde{U}_{q,\varphi}^{P}(\mathfrak{g})$ makes it clear that $\big(\widetilde{U}_{1,\varphi}^{P}(\mathfrak{g}) =\big) F[\mathcal{H}^\tau] \cong F[H^\tau]$ as Hopf algebras, hence $\mathcal{H}^\tau = H$ as algebraic groups (this is the trivial part, also in [DP], [DKP]). Moreover, the Hopf pairing among $\widehat{U}_{1,\varphi}^{Q}(\mathfrak{h}) \cong U(\mathfrak{h}^\tau)$ and $\widetilde{U}_{1,\varphi}^{P}(\mathfrak{g}) = F[\mathcal{H}^\tau] = F[H^\tau]$ is compatible with Poisson and co-Poisson structures, that is $\langle h, \{f,g\} \rangle = \langle \delta(h), f \otimes g \rangle$ for

all $h \in \widehat{U}_{1,\varphi}^Q(\mathfrak{h})$ and $f, g \in \widetilde{U}_{1,\varphi}^P(\mathfrak{g})$, where δ is the Poisson co-bracket of $\widehat{U}_1^Q(\mathfrak{h})$ and $\{\ ,\ \}$ can be either the Poisson bracket $\{\ ,\ \}_\star$ of $F[H^\tau]$ (giving the Poisson structure of H^τ) or the Poisson bracket $\{\ ,\ \}_\circ$ of $F[\mathcal{H}^\tau] = \widetilde{U}_{1,\varphi}^P(\mathfrak{g})$ (arising from its quantization $\widetilde{U}_{q,\varphi}^P(\mathfrak{g})$): since the pairing is perfect, we must have identity $\{\ ,\ \}_\star = \{\ ,\ \}_\circ$, whence the thesis. $\square$

8.4 The case $q \to 1$: specialization of $\widetilde{F}_{q,\varphi}^Q[G]$ and $\widetilde{F}_{q,\varphi}^Q[G]_{(q-1),\infty}$ to $U(\mathfrak{h})$. Consider the isomorphism $\xi_\varphi^Q \colon \widetilde{F}_{q,\varphi}^Q[G]_{(q-1),\infty} \xrightarrow{\cong} \widehat{U}_{q,\varphi}^Q(\mathfrak{h})_{(q-1),\infty} =: \widehat{U}_{q,\varphi}^Q(\mathfrak{h})$ provided by Theorem 7.2 *(b)*; specializing q to 1 we get a Hopf isomorphism (over k)

$$\left(\xi_\varphi^Q\right)_1 \colon \widetilde{F}_{q,\varphi}^Q[G]_{(q-1),\infty} \Big/ (q-1)\widetilde{F}_{q,\varphi}^Q[G]_{(q-1),\infty} \xrightarrow{\cong} \widehat{U}_{1,\varphi}^Q(\mathfrak{h})\ ; \tag{8.1}$$

in particular, we remark that now both $\widetilde{F}_{q,\varphi}^Q[G]_{(q-1),\infty} \Big/ (q-1)\widetilde{F}_{q,\varphi}^Q[G]_{(q-1),\infty}$ and $\widehat{U}_{1,\varphi}^Q(\mathfrak{h})$ are *non-formal* Hopf algebras; furthermore, the very definitions imply that

$$\widetilde{F}_{q,\varphi}^Q[G]_{(q-1),\infty} \Big/ (q-1)\widetilde{F}_{q,\varphi}^Q[G]_{(q-1),\infty} = \widetilde{F}_{1,\varphi}^Q[G]$$

where $\widetilde{F}_{1,\varphi}^Q[G] := \widetilde{F}_{q,\varphi}^Q[G] \Big/ (q-1)\widetilde{F}_{q,\varphi}^Q[G]$, hence (8.1) coincides with the specialization

$$\left(\xi_\varphi^Q\right)_1 \colon \widetilde{F}_{1,\varphi}^Q[G] := \widetilde{F}_{q,\varphi}^Q[G] \Big/ (q-1)\widetilde{F}_{q,\varphi}^Q[G] \hookrightarrow \widehat{U}_{1,\varphi}^Q(\mathfrak{h}) \tag{8.2}$$

of $\xi_\varphi^Q \colon \widetilde{F}_{q,\varphi}^Q[G] \hookrightarrow \widehat{U}_{q,\varphi}^Q(\mathfrak{h})$, thus also (8.2) is an isomorphism; clearly, both (8.1) and (8.2) are isomorphisms of Poisson Hopf coalgebras. Thus from $\widehat{U}_{1,\varphi}^Q(\mathfrak{h}) \cong U(\mathfrak{h}^\tau)$ we get:

Theorem 8.5. *For $q \to 1$, the Hopf algebra $\widetilde{F}_{q,\varphi}^Q[G]$ and the formal Hopf algebra $\widetilde{F}_{q,\varphi}^Q[G]_{(q-1),\infty}$ both specialize to the Poisson Hopf coalgebra $U(\mathfrak{h}^\tau)$.* $\square$

8.6 The case $q \to 1$: specialization of $\widehat{F}_{q,\varphi}^{P,\infty}[G]$ and $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h})$ to $F^\infty[G^\tau]$. We shall now show that $\widehat{F}_{q,\varphi}^{P,\infty}[G]$ and $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h})$ are (isomorphic) quantizations of $F^\infty[G^\tau]$; in particular $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h}) \xrightarrow{q \to 1} F^\infty[G^\tau]$, can be seen as the dual counterpart (in the sense of Poisson duality) of $\widetilde{U}_{q,\varphi}^P(\mathfrak{g}) \xrightarrow{q \to 1} F[H^\tau]$ (cf. (3.8)).

Theorem 8.7. *For $q \to 1$, the formal Hopf algebras $\widehat{F}_{q,\varphi}^{P,\infty}[G]$ and $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h})$ both specialize to the formal Poisson Hopf algebra $F^\infty[G^\tau]$.*

Proof. We stick for simplicity to the case $\varphi = 0$, generalization being straightforward.

We provide two proofs, the first one more geometric (a consequence of (4.6)), the second one more algebraic (a consequence of Theorem 8.2).

Geometric proof. Recall that $F^\infty[G]$ can be defined as the m_e-adic completion of $\mathcal{O}_e$ (where $(\mathcal{O}, m_e)$ denotes the local ring of $e \in G$); it is also clear that this coincides with the $\mathfrak{m}_e$-adic completion of $F[G]$, where $\mathfrak{m}_e$ is the maximal ideal of $e \in G$. On the other hand, $\widehat{F}_q^{P,\infty}[G]$ is the $\mathfrak{E}$-adic completion of $\widehat{F}_q^P[G]$, with $\mathfrak{E} := Ker\left(\varepsilon \colon \widehat{F}_q^P[G] \to k[q, q^{-1}]\right)$

(cf. §4.6); since $\widehat{F}_q^P[G] \xrightarrow{q \to 1} F[G]$ (cf. §4.5), $\widehat{F}_q^{P,\infty}[G]$ specializes to the $\mathfrak{E}_1$-adic completion of $F[G]$, with $\mathfrak{E}_1 := Ker(\varepsilon \colon F[G] \to k)$; but $\mathfrak{E}_1 = \mathfrak{m}_e$, whence the thesis.

Algebraic proof. Recall that $F^\infty[G]$ can be defined as the *full* linear dual of $U(\mathfrak{g})$, that is $F^\infty[G] := U(\mathfrak{g})^*$. From the proof of Lemma 5.3 we recall that

$$D_q^Q(\mathfrak{g})^* \cong \big(U_q^P(\mathfrak{b}_-) \widehat{\rtimes} k(q)[(k^\star)^n]\big) \,\widehat{\otimes}\, \big(U_q^P(\mathfrak{b}_+) \widehat{\rtimes} k(q)[(k^\star)^n]\big) \ ;$$

a quick review of the proof of Proposition 5.5 will then convince the reader that

$$U_q^Q(\mathfrak{g})^* \cong \left(pr^Q\right)^* \left(U_q^Q(\mathfrak{g})^*\right) = U_q(\mathfrak{n}_-) \,\widehat{\otimes}\, U_q^P(\mathfrak{t}) \,\widehat{\otimes}\, k(q)[{\mathbb{Z}_2}^n] \,\widehat{\otimes}\, U_q(\mathfrak{n}_+) \tag{8.3}$$

as $k(q)$-vector spaces (with notations of Theorem 5.7). But when looking at integer forms we have $\widehat{U}_q^Q(\mathfrak{t})^\circ = \widetilde{U}_q^P(\mathfrak{t})$, in short because non-scalar elements of $k(q)[(\mathbb{Z}_2)^n]$ take values outside $k[q,q^{-1}]$ when evaluated on $\widehat{U}_q^Q(\mathfrak{t})$: in fact $\left\langle \nu, \left({K_i;0 \atop 1}\right) \right\rangle = \frac{\nu_i - 1}{q_i - 1} \notin k[q,q^{-1}]$ unless $\nu_i = 1$ (for all $\nu \in {\mathbb{Z}_2}^n$, $i = 1, \dots, n$). Similarly we have $\widehat{U}_q^Q(\mathfrak{b}_+)^\circ = \widehat{F}_q^P[B_+] \cong \widetilde{U}_q^P(\mathfrak{b}_-)_{op}$, $\widehat{U}_q^Q(\mathfrak{b}_-)^\circ = \widehat{F}_q^P[B_-] \cong \widetilde{U}_q^P(\mathfrak{b}_+)_{op}$, $\widehat{U}_q^Q(\mathfrak{g})^\circ = \widehat{F}_q^P[G]$, so that (8.3) yields

$$\widehat{U}_q^Q(\mathfrak{g})^* \cong \left(pr^Q\right)^* \left(\widehat{U}_q^Q(\mathfrak{g})^*\right) = \widetilde{U}_q(\mathfrak{n}_-) \,\widehat{\otimes}\, \widetilde{U}_q^P(\mathfrak{t}) \,\widehat{\otimes}\, \widetilde{U}_q(\mathfrak{n}_+) \ ; \tag{8.4}$$

on the other hand, we record that $\widetilde{U}_q^{P,\infty}(\mathfrak{h}) \cong \left(\left(pr^Q\right)^* \circ \left(\xi_q^P\right)^{-1}\right)\left(\widetilde{U}_q^{P,\infty}(\mathfrak{h})\right) \cong$ $\cong \left(pr^Q\right)^* \left(\widehat{F}_q^{P,\infty}[G]\right) = \widehat{A}_\infty^P = \widetilde{U}_q(\mathfrak{n}_-) \,\widehat{\otimes}\, \widetilde{U}_q^P(\mathfrak{t}) \,\widehat{\otimes}\, \widetilde{U}_q(\mathfrak{n}_+)$ hence comparing with (8.4) we get

$$\widehat{U}_q^Q(\mathfrak{g})^* \cong \widetilde{U}_q^{P,\infty}(\mathfrak{h}) \ . \tag{8.5}$$

Now specialize q to 1: then $\widehat{U}_q^Q(\mathfrak{g})$ specializes to $U(\mathfrak{g})$; dualizing, specialization of q at 1 gives, in force of (8.5), $\widetilde{U}_1^{P,\infty}(\mathfrak{h}) := \widetilde{U}_q^{P,\infty}(\mathfrak{h}) \otimes_{k[q,q^{-1}]} k \cong \widehat{U}_q^Q(\mathfrak{g})^* \otimes_{k[q,q^{-1}]} k =$ $= \left(\widehat{U}_q^Q(\mathfrak{g}) \otimes_{k[q,q^{-1}]} k\right)^* = \widehat{U}_1^Q(\mathfrak{g})^* \cong U(\mathfrak{g})^*$, i. e.

$$\widetilde{U}_1^{P,\infty}(\mathfrak{h}) \cong U(\mathfrak{g})^* = F^\infty[G]$$

that is $\widetilde{U}_q^{P,\infty}(\mathfrak{h})$ specializes to $U(\mathfrak{g})^* = F^\infty[G]$ for $q \to$, q. e. d. $\square$

8.8 The case $q \to \varepsilon$: quantum Frobenius morphisms. Let ε be a primitive ℓ-th root of 1 in k, for ℓ *odd*, $\ell > d := \max_i\{d_i\}$, and set

$$\widehat{U}_{\varepsilon,\varphi}^Q(\mathfrak{h}) := \widehat{U}_{q,\varphi}^Q(\mathfrak{h}) \Big/ (q - \varepsilon) \widehat{U}_{q,\varphi}^Q(\mathfrak{h}) \cong \widehat{U}_{q,\varphi}^Q(\mathfrak{h}) \otimes_{k[q,q^{-1}]} k$$

(k being a $k[q,q^{-1}]$-algebra via $k \cong k[q,q^{-1}] \big/ (q - \varepsilon)$); we have the following result, which is the analog for $U_{q,\varphi}^{Q,\infty}(\mathfrak{h})$ of a similar result about $U_{q,\varphi}^Q(\mathfrak{g})$, which can be found in [CV-2], §3.2 (cf. also [Lu-2], Theorem 8.10, and [DL], Theorem 6.3):

Theorem 8.9. *There exists a (formal) Hopf algebra epimorphism*

$$\widehat{\mathcal{F}r}_{\mathfrak{h}^\tau} \colon \widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{h}) \twoheadrightarrow \widehat{U}^Q_{1,\varphi}(\mathfrak{h}) \cong U(\mathfrak{h}^\tau)$$

defined by

$$\begin{aligned} \widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}\left(\left. (F_i^\varphi)^{(s)} \right|_{q=\varepsilon} \right) &:= \left. (F_i^\varphi)^{(s/\ell)} \right|_{q=1} \qquad \text{if } \ell \Big| s\,,\ 0 \text{ otherwise} \\ \widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}\left(\left. \binom{K_i^\varphi;0}{s} \right|_{q=\varepsilon} \right) &:= \left. \binom{K_i^\varphi;0}{s/\ell} \right|_{q=1} \qquad \text{if } \ell \Big| s\,,\ 0 \text{ otherwise} \\ \widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}\left(\left. (K_i^\varphi)^{-1} \right|_{q=\varepsilon} \right) &:= 1 \\ \widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}\left(\left. (E_i^\varphi)^{(s)} \right|_{q=\varepsilon} \right) &:= \left. (E_i^\varphi)^{(s/\ell)} \right|_{q=1} \qquad \text{if } \ell \Big| s\,,\ 0 \text{ otherwise} \end{aligned}$$

for all $i = 1, \dots, n$, $s \in \mathbb{N}$. *Moreover,* $\widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}$ *is adjoint of* $\widetilde{\mathcal{F}r}_{\mathfrak{g}^\tau}$ *(cf. §3.5) with respect to the (specialized) quantum Poisson pairings, viz. (for all* $h \in \widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{h})$, $g \in \widetilde{U}^P_{1,\varphi}(\mathfrak{g})$ *)*

$$\pi^\varphi_{1,H^\tau}\left(\widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}(h), g \right) = \pi^\varphi_{\varepsilon,H^\tau}\left(h, \widetilde{\mathcal{F}r}_{\mathfrak{g}^\tau}(g) \right) .$$

Proof. Consider the Hopf algebra embedding $\widetilde{\mathcal{F}r}_{\mathfrak{g}^\tau} \colon F[H^\tau] \cong \widetilde{U}^P_{1,\varphi}(\mathfrak{g}) \hookrightarrow \widetilde{U}^P_{\varepsilon,\varphi}(\mathfrak{g})$; its (linear) dual is an epimorphism of formal Hopf algebras $\widetilde{U}^P_\varepsilon(\mathfrak{g})^* \twoheadrightarrow \widetilde{U}^P_1(\mathfrak{g})^*$; composing the latter with the natural embedding $\widehat{U}^Q_\varepsilon(\mathfrak{h}) \hookrightarrow \widetilde{U}^P_\varepsilon(\mathfrak{g})^*$ provided by the (specialized) quantum Poisson pairing $\pi^\varphi_{\varepsilon,H^\tau} \colon \widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{h}) \otimes \widetilde{U}^P_{\varepsilon,\varphi}(\mathfrak{g}) \longrightarrow k$ yields a morphism

$$\widehat{\mathcal{F}r}_{\mathfrak{h}^\tau} \colon \widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{h}) \longrightarrow \widetilde{U}^P_{1,\varphi}(\mathfrak{g})^* .$$

Then the very construction gives

$$\left\langle \widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}(h), g \right\rangle = \pi^\varphi_{1,H^\tau}\left(\widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}(h), g \right) = \pi^\varphi_{\varepsilon,H^\tau}\left(h, \widetilde{\mathcal{F}r}_{\mathfrak{g}^\tau}(g) \right)$$

for all $h \in \widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{h})$, $g \in \widetilde{U}^P_{1,\varphi}(\mathfrak{g})$, from which one immediately gets that $\widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}$ is described by the formulas above and has image $\widehat{U}^Q_{1,\varphi}(\mathfrak{h})$, q. e. d. $\square$

With similar arguments we also prove next result, which is the analog of (3.9); we set

$$\begin{aligned} \widetilde{U}^{P,\infty}_{1,\varphi}(\mathfrak{h}) &:= \widetilde{U}^{P,\infty}_{q,\varphi}(\mathfrak{h}) \Big/ (q-1)\widetilde{U}^{P,\infty}_{q,\varphi}(\mathfrak{h}) \;\cong\; \widetilde{U}^{P,\infty}_{q,\varphi}(\mathfrak{h}) \otimes_{k[q,q^{-1}]} k \\ \widetilde{U}^{P,\infty}_{\varepsilon,\varphi}(\mathfrak{h}) &:= \widetilde{U}^{P,\infty}_{q,\varphi}(\mathfrak{h}) \Big/ (q-\varepsilon)\widetilde{U}^{P,\infty}_{q,\varphi}(\mathfrak{h}) \;\cong\; \widetilde{U}^{P,\infty}_{q,\varphi}(\mathfrak{h}) \otimes_{k[q,q^{-1}]} k . \end{aligned}$$

Theorem 8.10.
(a) There exists a formal Hopf algebra monomorphism

$$\widetilde{\mathcal{F}r}_{\mathfrak{h}^\tau} \colon F^\infty[G^\tau] \cong \widetilde{U}^{P,\infty}_{1,\varphi}(\mathfrak{h}) \hookrightarrow \widetilde{U}^{P,\infty}_{\varepsilon,\varphi}(\mathfrak{h})$$

defined by

$$\widetilde{\mathcal{F}r}_{\mathfrak{h}^\tau} \colon \quad \overline{F}^\varphi_\alpha\Big|_{q=1} \mapsto \big(\overline{F}^\varphi_\alpha\big)^\ell\Big|_{q=\varepsilon}, \quad L^\varphi_\lambda\Big|_{q=1} \mapsto \big(L^\varphi_\lambda\big)^\ell\Big|_{q=\varepsilon}, \quad \overline{E}^\varphi_\alpha\Big|_{q=1} \mapsto \big(\overline{E}^\varphi_\alpha\big)^\ell\Big|_{q=\varepsilon}$$

for all $\alpha \in R^+$, $\lambda \in P$. Moreover, $\widetilde{\mathcal{F}r}_{\mathfrak{h}^\tau}$ is the continuos extension of $\widehat{\mathcal{F}r}_{G^\tau}$ (cf. §4.5) and is adjoint of $\widehat{\mathcal{F}r}_{\mathfrak{g}^\tau}$ (cf. §3.5) with respect to the (specialized) quantum Poisson pairings, viz. (for all $h \in \widetilde{U}^P_{1,\varphi}(\mathfrak{h})$, $g \in \widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{g})$)

$$\pi^\varphi_{\varepsilon,G^\tau}\left(\widetilde{\mathcal{F}r}_{\mathfrak{h}^\tau}(h), g\right) = \pi^\varphi_{1,G^\tau}\left(h, \widehat{\mathcal{F}r}_{\mathfrak{g}^\tau}(g)\right).$$

(b) The image $Z_0 \left(\cong_{\widetilde{\mathcal{F}r}_{\mathfrak{h}^\tau}} \widetilde{U}^{P,\infty}_{1,\varphi}(\mathfrak{h}) \right)$ of $\widetilde{\mathcal{F}r}_{\mathfrak{h}^\tau}$ is a central formal Hopf subalgebra of $\widetilde{U}^{P,\infty}_{\varepsilon,\varphi}(\mathfrak{h})$.

(c) The set of ordered PBW monomials

$$\left\{ \prod_{r=N}^{1} \left(\overline{F}^\varphi_{\alpha^r}\right)^{\ell f_r} \cdot \prod_{i=1}^{n} \big(L^\varphi_i\big)^{\ell l_i} \cdot \prod_{r=1}^{N} \left(\overline{E}^\varphi_{\alpha^r}\right)^{\ell e_r} \,\middle|\, f_1, \dots, f_N, l_1, \dots, l_n, e_1, \dots, e_N \in \mathbb{N} \right\}$$

is a pseudobasis of Z_0 (over k).

(d) The set of ordered PBW monomials

$$\left\{ \prod_{r=N}^{1} \left(\overline{F}^\varphi_{\alpha^r}\right)^{e_r} \cdot \prod_{i=1}^{n} \big(L^\varphi_i\big)^{l_i} \cdot \prod_{s=1}^{N} \left(\overline{E}^\varphi_{\alpha^s}\right)^{e_s} \,\middle|\, f_r, l_i, e_s = 0, 1, \dots, \ell-1 \;\forall\, r, i, s \right\}$$

is a basis of $\widetilde{U}^{P,\infty}_{\varepsilon,\varphi}(\mathfrak{h})$ over Z_0 ; thus $\widetilde{U}^{P,\infty}_{\varepsilon,\varphi}(\mathfrak{h})$ is a free module of rank $\ell^{dim(H)}$ over Z_0 .

Proof. As for *(a)*, consider the Hopf algebra epimorphism $\widehat{\mathcal{F}r}_{\mathfrak{g}^\tau} \colon \widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{g}) \twoheadrightarrow \widehat{U}^Q_{1,\varphi}(\mathfrak{g}) \cong U(\mathfrak{g}^\tau)$; its (linear) dual is a monomorphism of formal Hopf algebras $\widehat{U}^Q_{1,\varphi}(\mathfrak{g})^* \hookrightarrow \widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{g})^*$; composing the latter with the natural embedding $\widetilde{U}^{P,\infty}_{1,\varphi}(\mathfrak{h}) \hookrightarrow \widehat{U}^Q_{1,\varphi}(\mathfrak{g})^*$ provided by the (specialized) quantum Poisson pairing $\pi^\varphi_{1,G^\tau} \colon \widetilde{U}^{P,\infty}_{1,\varphi}(\mathfrak{h}) \otimes \widehat{U}^Q_{1,\varphi}(\mathfrak{g}) \longrightarrow k$ yields a morphism

$$\widetilde{\mathcal{F}r}_{\mathfrak{h}^\tau} \colon \widetilde{U}^{P,\infty}_{1,\varphi}(\mathfrak{h}) \longrightarrow \widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{g})^*.$$

Thus the very construction gives

$$\left\langle \widetilde{\mathcal{F}r}_{\mathfrak{h}^\tau}(h), g \right\rangle = \pi^\varphi_{\varepsilon,G^\tau}\left(\widetilde{\mathcal{F}r}_{\mathfrak{h}}(h), g\right) = \pi^\varphi_{1,G^\tau}\left(h, \widehat{\mathcal{F}r}_{\mathfrak{g}}(g)\right)$$

for all $h \in \widetilde{U}^P_{1,\varphi}(\mathfrak{h})$, $x \in \widehat{U}^Q_\varepsilon(\mathfrak{g})$: then from the definition of π^φ_{q,G^τ} one immediately gets that $\widetilde{\mathcal{F}r}_{\mathfrak{h}^\tau}$ is described by the formulas above, hence in particular its image is contained in $\widetilde{U}^P_{\varepsilon,\varphi}(\mathfrak{h})$ (embedded in $\widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{g})^*$ by means of $\pi^\varphi_{\varepsilon,G^\tau}$). Finally, since $\widehat{\mathcal{F}r}_{G^\tau} \colon F[G^\tau] \cong \widehat{F}^P_{1,\varphi}[G] \hookrightarrow \widehat{F}^P_{\varepsilon,\varphi}[G]$ is also defined as (Hopf) dual of $\widehat{\mathcal{F}r}_{\mathfrak{g}^\tau} \colon \widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{g}) \twoheadrightarrow \widehat{U}^Q_{1,\varphi}(\mathfrak{g}) \cong U(\mathfrak{g}^\tau)$ (cf. [DL], Proposition 6.4, along with [CV-2], §3), then $\widetilde{\mathcal{F}r}_{\mathfrak{h}^\tau} \colon F^\infty[G^\tau] \cong \widetilde{U}^{P,\infty}_{1,\varphi}(\mathfrak{h}) \hookrightarrow \widetilde{U}^{P,\infty}_{\varepsilon,\varphi}(\mathfrak{h})$ is obviously an extension of $\widehat{\mathcal{F}r}_{G^\tau} \colon F[G^\tau] \cong \widehat{F}^P_{1,\varphi}[G] \hookrightarrow \widehat{F}^P_{\varepsilon,\varphi}[G]$, and it is also clear that this extension is by continuity.

Parts *(b)*, *(c)*, and *(d)* can be easily deduced from the analogous result for $\widetilde{U}^P_\varepsilon(\mathfrak{g})$ (namely [DP], Theorem 19.1 and its generalization) along with the very definitions of $U^P_{q,\varphi}(\mathfrak{g})$ and $U^{P,\infty}_{q,\varphi}(\mathfrak{h})$ (otherwise one can directly use computations in [DP], §19, or mimick the proof of Proposition 6.4 of [DL]). $\square$

By the way, we remark that part *(d)* of Theorem 8.10 is well-related with [CV-2], Proposition 3.5 (extending [DL], Theorem 7.2), which claims that $\widehat{F}^P_{\varepsilon,\varphi}[G]$ is a projective module over F_0 of rank $\ell^{dim(G^\tau)} = \ell^{dim(H^\tau)}$.

At last, we can prove the following counterpart of [CV-2], §3.3 (cf. also [DL], Proposition 6.4 for the one-parameter case), which dealt with (4.6): notice in particular that now we get a *surjective* morphism instead of an *injective* one (like is in [CV-2] and [DL]).

Theorem 8.11. *There exists a Hopf algebra epimorphism*

$$\widetilde{\mathcal{F}r}_{H^\tau} \colon \widetilde{F}^Q_{\varepsilon,\varphi}[G] \twoheadrightarrow \widetilde{F}^Q_{1,\varphi}[G] \cong U(\mathfrak{h}^\tau)$$

dual of $\widetilde{\mathcal{F}r}_{\mathfrak{g}^\tau} \colon F[H^\tau] \cong \widetilde{U}^P_{1,\varphi}(\mathfrak{g}) \hookrightarrow \widetilde{U}^P_{\varepsilon,\varphi}(\mathfrak{g})$.

Proof. Since $\widetilde{F}^Q_{\varepsilon,\varphi}[G] \leq \widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{h})$, we can restrict $\widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}$ to $\widetilde{F}^Q_{\varepsilon,\varphi}[G]$, thus getting a Hopf algebra morphism $\widetilde{\mathcal{F}r}_{H^\tau} \colon \widetilde{F}^Q_{\varepsilon,\varphi}[G] \longrightarrow \widehat{U}^Q_{1,\varphi}(\mathfrak{h}) \cong U(\mathfrak{h}^\tau)$. Now Theorem 5.13 *(b1)–(b2)* gives (adapting notations to the multiparameter setting)

$$\widetilde{\mathcal{F}r}_{H^\tau}\left(\widetilde{F}^Q_{\varepsilon,\varphi}[G]\right) = \widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}\left(\widetilde{F}^Q_{\varepsilon,\varphi}[G]\left[\psi^{-1}_{-\delta}\big|_{q=\varepsilon}\right]\right) = \widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}\left(\widetilde{A}^{\varphi,Q}\big|_{q=\varepsilon}\right)$$

because[6] $\psi^{-1}_{-\delta} \cong \mu^Q_\varphi(\psi_{-\delta}) = K^\varphi_{-\delta} = \prod_{i=1}^n (K^\varphi_i)^{-1}$ whence $\widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}\left(\psi^{-1}_{-\delta}\big|_{q=\varepsilon}\right) = = \prod_{i=1}^n \widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}\left((K^\varphi_i)^{-1}\big|_{q=\varepsilon}\right) = 1$; moreover, it is immediate from definitions that

$$\widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}\left(\widetilde{A}^{\varphi,Q}\big|_{q=\varepsilon}\right) = \widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}\left(\widehat{U}^Q_{\varepsilon,\varphi}(\mathfrak{h})\right) = \widehat{U}^Q_{1,\varphi}(\mathfrak{h}) = \widetilde{F}^Q_{1,\varphi}[G] \ \ (\cong U(\mathfrak{h}^\tau))$$

hence we conclude that $\widetilde{\mathcal{F}r}_{H^\tau}\left(\widetilde{F}^Q_{\varepsilon,\varphi}[G]\right) = \widetilde{F}^Q_{1,\varphi}[G]$; the thesis follows. $\square$

[6] Notice that the matrix coefficient $\psi_{-\delta}$ does not depend on φ, but its image $\mu^Q_\varphi(\psi_{-\delta}) \in \widetilde{A}^{\varphi,Q}$ does.

Like in the case of $\mathfrak{g}$, we call also $\widehat{\mathcal{F}r}_{\mathfrak{h}^\tau}$, $\widetilde{\mathcal{F}r}_{\mathfrak{h}^\tau}$, and $\widetilde{\mathcal{F}r}_{H^\tau}$ **quantum Frobenius morphisms**, because for $\ell = p$ prime they can be thought of as lifting of classical Frobenius morphisms $H^\tau_{\mathbb{Z}_p} \longrightarrow H^\tau_{\mathbb{Z}_p}$, $G^\tau_{\mathbb{Z}_p} \longrightarrow G^\tau_{\mathbb{Z}_p}$ to characteristic zero[7].

8.12 Specializing the quantum Poisson pairing. From the analysis in §§8.2–7 it follows that the Hopf pairing $\pi^\varphi_{q,H^\tau} \colon \widehat{U}^{Q,\infty}_{q,\varphi}(\mathfrak{h}) \otimes \widetilde{U}^P_{q,\varphi}(\mathfrak{g}) \to k[q,q^{-1}]$, resp. $\pi^\varphi_{q,G^\tau} \colon \widetilde{U}^{P,\infty}_{q,\varphi}(\mathfrak{h}) \otimes \widehat{U}^Q_{q,\varphi}(\mathfrak{g}) \to k[q,q^{-1}]$ specializes to the natural Hopf pairing $\pi_{H^\tau} \colon U(\mathfrak{h}^\tau) \otimes F[H^\tau] \to k$, resp. $\pi_{G^\tau} \colon F^\infty[G^\tau] \otimes U(\mathfrak{g}^\tau) \to k$ given by evaluation: thus the quantum Poisson pairing can be seen as a quantization of the previous pairings at the classical level. Now we show that it can also be seen as a quantization of the classical Poisson pairing $\pi^\tau_{\mathcal{P}} \colon \mathfrak{h}^\tau \otimes \mathfrak{g}^\tau \to k$.

To begin with, we define a grading on $U^{Q,\infty}_{q,\varphi}(\mathfrak{h})$ by giving to PBW monomials the degree

$$\partial \left(\prod_{r=N}^{1} (F^\varphi_{\alpha^r})^{(m_r)} \cdot \prod_{i=1}^{n} \binom{K^\varphi_i ; 0}{t_i} \cdot \prod_{r=1}^{N} (E^\varphi_{\alpha^r})^{(n_r)} \right) := \sum_{r=N}^{1} m_r + \sum_{i=1}^{n} t_i + \sum_{r=1}^{N} n_r$$

and extending by linearity (this is a grading as a $k(q)$-vector space). Then we set

$$\pi^\varphi_{q,\mathcal{P}}(h,g) := \, (q-1)^{\partial(h)} \cdot \pi^\varphi_q(h,g)$$

for all tensors $h \otimes g$ of PBW monomials in $\widehat{U}^Q_{q,\varphi}(\mathfrak{h}) \otimes \widehat{U}^Q_{q,\varphi}(\mathfrak{g})$ $\big(\subseteq U^{P,\infty}_{q,\varphi}(\mathfrak{h}) \otimes U^Q_{q,\varphi}(\mathfrak{g}) \big)$, and finally extend by linearity: this gives a perfect (linear) pairing $\pi^\varphi_{q,\mathcal{P}} \colon U^{Q,\infty}_{q,\varphi}(\mathfrak{h}) \otimes U^Q_{q,\varphi}(\mathfrak{g}) \to k(q)$ such that $\pi^\varphi_{q,\mathcal{P}} \Big(\widehat{U}^Q_{q,\varphi}(\mathfrak{h}), \widehat{U}^Q_{q,\varphi}(\mathfrak{g}) \Big) \subseteq k[q,q^{-1}]_{(q-1)}$ (where $k[q,q^{-1}]_{(q-1)}$ is the localization of $k[q,q^{-1}]$ at the principal prime ideal $(q-1)$), hence (by restriction) a pairing

$$\pi^\varphi_{q,\mathcal{P}} \colon \widehat{U}^Q_{q,\varphi}(\mathfrak{h}) \otimes \widehat{U}^Q_{q,\varphi}(\mathfrak{g}) \to k[q,q^{-1}]_{(q-1)}$$

which can be specialized at $q=1$.

Theorem 8.13. *The pairing $\pi^\varphi_{q,\mathcal{P}} \colon \widehat{U}^Q_{q,\varphi}(\mathfrak{h}) \otimes \widehat{U}^Q_{q,\varphi}(\mathfrak{g}) \to k[q,q^{-1}]_{(q-1)}$ specializes to a perfect linear pairing*

$$\pi^\tau_{\mathcal{P}} \colon U(\mathfrak{h}^\tau) \otimes U(\mathfrak{g}^\tau) \to k$$

which extends the Lie bialgebra pairing $\pi^\varphi_{\mathcal{P}} \colon \mathfrak{h}^\tau \otimes \mathfrak{g}^\tau \to k$ (cf. §1.2).

Proof. It is clearly enough to check that $\pi^\varphi_{q,\mathcal{P}}(h,g)\big|_{q=1} = \pi^\tau_{\mathcal{P}} \Big(h\big|_{q=1}, g\big|_{q=1} \Big)$ for Chevalley generators and root vectors $\mathrm{h}^\tau = h\big|_{q=1}$ of $\mathfrak{h}^\tau$ and $\mathrm{g} = g\big|_{q=1}$ of $\mathfrak{g}^\tau$, viz. for elements $F^\varphi_\alpha\big|_{q=1} = \mathrm{f}^\tau_\alpha$, $\left. \binom{K^\varphi_i ; 0}{1} \right|_{q=1} = \mathrm{h}^\tau_i$, $E^\varphi_\alpha\big|_{q=1} = \mathrm{e}^\tau_\alpha$ for $\widehat{U}^Q_{q,\varphi}(\mathfrak{h})$ specializing to $U(\mathfrak{h}^\tau)$, and $F_\alpha\big|_{q=1} = f_\alpha$, $\left. \binom{K_i ; 0}{1} \right|_{q=1} = h_i$, $E_\alpha\big|_{q=1} = e_\alpha$ for $\widehat{U}^Q_q(\mathfrak{g})$ specializing to $U(\mathfrak{g})$. The thesis then follows from direct (and straightforward) computation, e. g. (cf. (1.1–a/b))

$$\pi^\varphi_{q,\mathcal{P}}(F^\varphi_\alpha, E_\beta)\big|_{q=1} = (q-1) \left. \frac{-\delta_{\alpha\beta}}{(q_\alpha - q_\alpha^{-1})} \right|_{q=1} = -\frac{1}{2} \delta_{\alpha\beta} d_\alpha^{-1} = \pi^\tau_{\mathcal{P}}(\mathrm{f}^\tau_\alpha, e_\beta) \, . \, \square$$

[7] Here again $H^\tau_{\mathbb{Z}_p}$ denotes the Chevalley-type Poisson group-scheme over $\mathbb{Z}_p$ that one can clearly build up from the "Cartan datum" associated to H^τ (mimicking the usual procedure one follows for $G_{\mathbb{Z}_p}$); as a group-scheme it is isomorphic to $H_{\mathbb{Z}_p} := H^0_{\mathbb{Z}_p}$, the difference residing only in the Poisson structure.

8.14 The pairings $F[G^\tau] \otimes F[H^\tau] \longrightarrow k$**,** $F^\infty[G^\tau] \otimes F[H^\tau] \longrightarrow k$**.** The construction of §8.12 can be reversed as follows; we define a pairing $\pi_q^{\varphi,\mathcal{P}} \colon \widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h}) \otimes \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) \to k[q,q^{-1}]$ modifying π_q^φ: give to PBW monomials of $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h})$ or $\widetilde{U}_{q,\varphi}^{P}(\mathfrak{g})$ the degree

$$\partial \left(\prod_{r=N}^{1} \left(\overline{F}_{\alpha^r}^{\varphi}\right)^{m_r} \cdot \prod_{i=1}^{n} (L_i^\varphi)^{l_i} \cdot \prod_{r=1}^{N} \left(\overline{E}_{\alpha^r}^{\varphi}\right)^{n_r} \right) := \sum_{r=N}^{1} m_r + \sum_{i=1}^{N} n_r$$

and extend by linearity to define a (linear) grading of $U_{q,\varphi}^{P,\infty}(\mathfrak{h})$ or $U_{q,\varphi}^{P}(\mathfrak{g})$. Now extend $\pi_q^\varphi \colon U_{q,\varphi}^{P,\infty}(\mathfrak{h}) \otimes U_{q,\varphi}^{Q}(\mathfrak{g}) \to k(q)$ (or $\pi_q^\varphi \colon U_{q,\varphi}^{Q,\infty}(\mathfrak{h}) \otimes U_{q,\varphi}^{P}(\mathfrak{g}) \to k(q)$) to a perfect pairing $\pi_q^\varphi \colon U_{q,\varphi}^{P,\infty}(\mathfrak{h}) \otimes U_{q,\varphi}^{P}(\mathfrak{g}) \to k\left(q^{1/d}, q^{-1/d}\right)$ ($d := det\,(a_{ij})_{i,j=1,\dots,n}$) by $\pi_q^\varphi(L_\lambda^\varphi, L_\mu) := q^{(\lambda|\mu)}$. Then set

$$\pi_q^{\varphi,\mathcal{P}}(h,g) := \, (q-1)^{-\partial(h)} \cdot \pi_q^\varphi(h,g) = (q-1)^{-\partial(g)} \cdot \pi_q^\varphi(h,g)$$

for all $h \in \widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h})$ and $g \in \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g})$ *homogeneous*, with $\partial(x) :=$ degree of x, and extend by linearity to all of $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h}) \otimes \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g})$: then $\pi_q^{\varphi,\mathcal{P}}\left(\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h}), \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g})\right) \subseteq k\big[q^{1/d}, q^{-1/d}\big]$, thus we get perfect pairings

$$\pi_q^{\varphi,\mathcal{P}} \colon \widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h}) \otimes \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) = \widehat{F}_{q,\varphi}^{P,\infty}[G] \otimes \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) \longrightarrow k\left[q^{1/d}, q^{-1/d}\right]$$

$$\pi_q^{\varphi,\mathcal{P}} \colon \widehat{F}_{q,\varphi}^{P}[G] \otimes \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) \longrightarrow k\left[q^{1/d}, q^{-1/d}\right]$$

(the latter arising by restricting the former): these can be specialized at $q^{1/d} = 1$, yielding

Theorem 8.15. *The perfect pairings* $\pi_q^{\varphi,\mathcal{P}} \colon \widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h}) \otimes \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) \to k\left[q^{1/d}, q^{-1/d}\right]$ *and* $\pi_q^{\varphi,\mathcal{P}} \colon \widehat{F}_{q,\varphi}^{P}[G] \otimes \widetilde{U}_{q,\varphi}^{P}(\mathfrak{g}) \to k\left[q^{1/d}, q^{-1/d}\right]$ *specialize to perfect pairings*

$$\pi^{\tau,\mathcal{P}} \colon F^\infty[G^\tau] \otimes F[H^\tau] \to k\,, \quad \pi^{\tau,\mathcal{P}} \colon F[G^\tau] \otimes F[H^\tau] \to k\,. \;\square$$

8.16 Final remarks. *(a)* Most of our results — as far as most of the results we quoted from [DL], [DP], [CV-1/2], etc. — are in fact available also for quantum (enveloping/function) algebras with any weight lattice M (with $Q \leq M \leq P$), e. g. $U_{q,\varphi}^{M,\infty}(\mathfrak{h})$, $F_{q,\varphi}^{M}[G]$, etc.

(b) The specialization $\widetilde{U}_{q,\varphi}^{P,\infty}(\mathfrak{h}) \xrightarrow{q \to 1} F^\infty[G^\tau]$ (Theorem 8.7) can be seen as a particular concrete realization of the equivalence among the category of *quantum universal enveloping algebras* and the category of *quantum formal series Hopf algebras* described in [Dr], §7.

(c) Constructions and results about integer forms have been formulated with $k[q,q^{-1}]$ as ground ring for simplicity; nevertheless, one could check that same forms can in fact be defined (and results proved) also on the smaller ring $\mathbb{Z}[q,q^{-1}]$, exactly like it occurs for integer forms of $U_{q,\varphi}^{M}(\mathfrak{g})$ and $F_{q,\varphi}^{M}[G]$ (cf. [DL] and [CV-2]).

Appendix: the case $G = SL(2,k)$

A.1. The present appendix is devoted to the particularly simple case of $G = SL(2,k)$: we will exhibit some explicit formulas by direct computation — following the general pattern described in the main text — and then relate this approach with the well known technique of studying $F_q[SL(2,k)]$ by generators and relations; it will then be clear that such a comparison can be done more in general for the group $G = SL(n+1,k)$ too.

A.2. From §3 we recall that $D_q^P(\mathfrak{g}) = U_q^P(\mathfrak{b}_+) \otimes U_q^Q(\mathfrak{b}_-)$ (resp. $D_q^Q(\mathfrak{g}) = U_q^Q(\mathfrak{b}_+) \otimes$ $\otimes U_q^Q(\mathfrak{b}_-)$) is generated by $E \otimes 1$, $L \otimes 1$ (resp. $K \otimes 1$), $1 \otimes K$, $1 \otimes F$, with defining relations (3.2) (where $E = E \otimes 1$, and so on). Using these and relations (3.3) we find the following "straightening laws" for the product of monomials of a PBW basis of $D_q^P(\mathfrak{g})$

$$\left(\overline{E}^r L^\ell \otimes K^k \overline{F}^s\right) \cdot \left(\overline{E}^{r'} L^{\ell'} \otimes K^{k'} \overline{F}^{s'}\right) = \\ = \sum_{t \geq 0}^{t \leq r,s} q^{(\ell+2k)(r'-t)+(\ell'+2k')(s-t)} \cdot \begin{bmatrix} r' \\ t \end{bmatrix}_q \cdot \begin{bmatrix} s \\ t \end{bmatrix}_q \cdot [t]_q!^2 \cdot \left(q-q^{-1}\right)^{2t} \cdot \\ \cdot \overline{E}^{r+r'-t} L^{\ell+\ell'} \cdot \begin{bmatrix} K^{-1 \otimes}_{\ \otimes}; 2t-r'-s \\ t \end{bmatrix}_q \cdot K^{k+k'} \cdot \overline{F}^{s+s'-t} \tag{A.1}$$

and for the product of monomials of a PBW basis of $D_q^Q(\mathfrak{g})$

$$\left(E^{(r)} K^h \otimes K^k F^{(s)}\right) \cdot \left(E^{(r')} K^{h'} \otimes K^{k'} F^{(s')}\right) = \\ = \sum_{t \geq 0}^{t \leq r,s} q^{2((h+k)(r'-t)+(h'+k')(s-t))} \cdot \begin{bmatrix} r+r'-t \\ r \end{bmatrix}_q \cdot \begin{bmatrix} s+s'-t \\ s' \end{bmatrix}_q \cdot \\ \cdot E^{(r+r'-t)} K^{h+h'} \cdot \begin{bmatrix} K^{-1 \otimes}_{\ \otimes}; 2t-r'-s \\ t \end{bmatrix}_q \cdot K^{k+k'} F^{(s+s'-t)} . \tag{A.2}$$

A.3. Following §5, we identify the $k(q)$-algebra $U_q^M(\mathfrak{b}_-)_{op} \otimes U_q^P(\mathfrak{b}_+)_{op}$ with a subalgebra of $D_q^{M'}(\mathfrak{g})^*$, and denote by $F_q^M[D] := U_q^M(\mathfrak{b}_-)_{op} \widehat{\otimes}\, U_q^P(\mathfrak{b}_+)_{op}$ its completion; we shall now describe by explicit formulas the formal Hopf algebra structure of $A_\infty^{0,M}$ — for $M = P, Q$, by restriction of that of $F_q^M[D]$. We already know that

$$\varepsilon(F \otimes 1) = 0\,, \quad \varepsilon(L^{\pm 1} \otimes 1) = 1\,, \quad \varepsilon(1 \otimes L^{\pm 1}) = 1\,, \quad \varepsilon(1 \otimes E) = 0$$
$$\text{resp.} \quad \varepsilon(F \otimes 1) = 0\,, \quad \varepsilon(K^{\pm 1} \otimes 1) = 1\,, \quad \varepsilon(1 \otimes L^{\pm 1}) = 1\,, \quad \varepsilon(1 \otimes E) = 0$$

thus the counity of $F_q^P[D]$, resp. $F_q^Q[D]$, hence of $A_\infty^{0,P}$, resp. $A_\infty^{0,Q}$, is completely determined. As for comultiplication $\Delta_{D_q^{M'}(\mathfrak{g})^*} := \left(m_{D_q^{M'}(\mathfrak{g})}\right)^* : D_q^{M'}(\mathfrak{g})^* \to D_q^{M'}(\mathfrak{g})^* \,\widehat{\otimes}\, D_q^{M'}(\mathfrak{g})^*$, we perform direct computation. We begin with $\overline{F} \otimes 1 \in D_q^Q(\mathfrak{g})^*$: we have

$$\left\langle \Delta\left(\overline{F} \otimes 1\right), \left(E^{(r)} K^h \otimes K^k F^{(s)}\right) \otimes \left(E^{(r')} K^{h'} \otimes K^{k'} F^{(s')}\right) \right\rangle = \\ = \langle \overline{F} \otimes 1, \left(E^{(r)} K^h \otimes K^k F^{(s)}\right) \cdot \left(E^{(r')} K^{h'} \otimes K^{k'} F^{(s')}\right) =$$

$$= \sum_{t\geq 0}^{t\leq r',s} q^{2((h+k)(r'-t)+(h'+k')(s-t))} \cdot \begin{bmatrix} r+r'-t \\ r \end{bmatrix}_q \cdot \begin{bmatrix} s+s'-t \\ s' \end{bmatrix}_q \cdot$$

$$\cdot \left\langle \overline{F} \otimes 1, E^{(r+r'-t)} K^{h+h'} \cdot \begin{bmatrix} K^{-1}{}^{\otimes}_{\otimes}; 2t-r'-s \\ t \end{bmatrix}_q \cdot K^{k+k'} F^{(s+s'-t)} \right\rangle =$$

$$= \delta_{s',0} \cdot \delta_{r+r'-s,1} \cdot q^{2(h+k)(r'-s)} \cdot \begin{bmatrix} 1 \\ r \end{bmatrix}_q \cdot \left\langle \overline{F} \otimes 1, E^{(1)} K^{h+h'} \cdot \begin{bmatrix} K^{-1}{}^{\otimes}_{\otimes}; s-r' \\ s \end{bmatrix}_q \cdot K^{k+k'} \right\rangle =$$

$$= \delta_{s',0} \cdot (-1)^{s+1} \cdot \left(\delta_{r,0} \cdot \delta_{r',s+1} \cdot q^{2(h+k)} + \delta_{r,1} \cdot \delta_{r',s} \cdot \delta_{s,0} \right) \ ;$$

now the element $\left(\overline{F} \otimes 1\right) \otimes (1 \otimes 1) + \sum_{n=0}^{\infty} q^{-n} \cdot \left(K^{-1} \otimes K \overline{E}^n\right) \otimes \left(\overline{F}^{n+1} \otimes 1\right)$ $\left(\in F_q^P[D]^{\otimes 2}\right)$ takes on PBW monomials $\left(E^{(r)} K^h \otimes K^k F^{(s)}\right) \otimes \left(E^{(r')} K^{h'} \otimes K^{k'} F^{(s')}\right)$ the same values as $\Delta\left(\overline{F} \otimes 1\right)$, hence

$$\Delta\left(\overline{F} \otimes 1\right) = \left(\overline{F} \otimes 1\right) \otimes (1 \otimes 1) + \sum_{n=0}^{\infty} q^{-n} \cdot \left(K^{-1} \otimes K \overline{E}^n\right) \otimes \left(\overline{F}^{n+1} \otimes 1\right) \ .$$

With exactly the same technique we get formulas

$$\begin{aligned} \Delta\left(L^{-1} \otimes L\right) &= \sum_{n=0}^{\infty} \left(L^{-1} \otimes L \overline{E}^n\right) \otimes \left(\overline{F}^n L^{-1} \otimes L\right) \\ \Delta\left(L \otimes L^{-1}\right) &= \left(L \otimes L^{-1}\right) \otimes \left(L \otimes L^{-1}\right) - \left(L \otimes L^{-1} \overline{E}\right) \otimes \left(\overline{F} L \otimes L^{-1}\right) \\ \Delta\left(1 \otimes \overline{E}\right) &= (1 \otimes 1) \otimes \left(1 \otimes \overline{E}\right) + \sum_{n=0}^{\infty} q^{+n} \cdot \left(1 \otimes \overline{E}^{n+1}\right) \otimes \left(\overline{F}^n K^{-1} \otimes K\right) \ ; \end{aligned}$$

by the way, this also proves $\Delta\left(A_\infty^{0,P}\right) \leq A_\infty^{0,P} \,\widehat{\otimes}\, A_\infty^{0,P}$ and[8] $\Delta\left(\widetilde{A}_\infty^{0,P}\right) \leq \widetilde{A}_\infty^{0,P} \,\widehat{\otimes}\, \widetilde{A}_\infty^{0,P}$.

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[8] In fact one can also compute (in the same way) the values $\Delta\left(L^{\pm 1} \otimes 1\right)$, $\Delta\left(1 \otimes L^{\pm 1}\right)$, and so directly verify that $\Delta\left(F_q^P[D]\right) \leq F_q^P[D] \,\widehat{\otimes}\, F_q^P[D]$.

The same procedure for $A_\infty^{0,Q}$ — using formula (A.2) — gives

$$\Delta\left(F^{(1)}\otimes 1\right)=\left(F^{(1)}\otimes 1\right)\otimes(1\otimes 1)+$$
$$+\sum_{n=0}^{\infty} q^{-n}\cdot\left(q-q^{-1}\right)^{2n}\cdot[n]_q!\cdot[n+1]_q!\cdot\left(K^{-1}\otimes K\overline{E}^n\right)\otimes\left(\overline{F}^{n+1}\otimes 1\right)$$
$$\Delta\left(K^{-1}\otimes K\right)=\sum_{n=0}^{\infty}\left(q-q^{-1}\right)^{2n}\cdot[n]_q!^2\cdot[n+1]_q\cdot\left(K^{-1}\otimes KE^{(n)}\right)\otimes\left(F^{(n)}K^{-1}\otimes K\right)$$
$$\Delta\left(K\otimes K^{-1}\right)=\left(K\otimes K^{-1}\right)\otimes\left(K\otimes K^{-1}\right)-$$
$$-\left(q-q^{-1}\right)^2\cdot[2]_q\cdot\left(K\otimes K^{-1}E^{(1)}\right)\otimes\left(F^{(1)}K\otimes K^{-1}\right)+$$
$$+\left(q-q^{-1}\right)^4\cdot{[2]_q}^2\cdot\left(K\otimes K^{-1}E^{(2)}\right)\otimes\left(F^{(2)}K\otimes K^{-1}\right)$$
$$\Delta\left(1\otimes E^{(1)}\right)=(1\otimes 1)\otimes\left(1\otimes E^{(1)}\right)+$$
$$+\sum_{n=0}^{\infty} q^{+n}\cdot\left(q-q^{-1}\right)^{2n}\cdot[n+1]_q!\cdot[n]_q!\cdot\left(1\otimes E^{(n+1)}\right)\otimes\left(F^{(n)}K^{-1}\otimes K\right)\ ;$$

notice that, as explained in the main text (§5.15), we deal with series which are convergent in the $(q-1)$-adic topology; moreover, formulas above explicitly show that $\Delta\left(A_\infty^{0,Q}\right)\leq$ $\leq A_\infty^{0,Q}\,\widehat{\otimes}\,A_\infty^{0,Q}$ and[9] $\Delta\left(\widehat{A}_\infty^{0,Q}\right)\leq\widehat{A}_\infty^{0,Q}\,\widehat{\otimes}\,\widehat{A}_\infty^{0,Q}$.

Similarly we proceed with the antipode; recall that $S_{D_q^{M'}(\mathfrak{g})^*}$ is defined as $S_{D_q^{M'}(\mathfrak{g})^*}:=$ $\left(S_{D_q^{M'}(\mathfrak{g})}\right)^*\colon D_q^{M'}(\mathfrak{g})^*\to D_q^{M'}(\mathfrak{g})^*$; then direct computation gives

$$\left\langle S\left(\overline{F}\otimes 1\right),\left(E^{(r)}K^h\otimes K^kF^{(s)}\right)\right\rangle=\left\langle\overline{F}\otimes 1,S\left(E^{(r)}K^h\otimes K^kF^{(s)}\right)\right\rangle=$$
$$=(-1)^{r+s}\cdot q^{-s(s-1)-r(r+1)-2rh-2rk+2rs}\cdot$$
$$\cdot\sum_{t\geq 0}^{t\leq r,s} q^{2((r-h)+(s-k))(s-t)}\cdot\left\langle\overline{F}\otimes 1,E^{(r-t)}\cdot K^{r-h}\begin{bmatrix}K^{-1}{}^{\otimes}_{\otimes};2t-r-s\\ t\end{bmatrix}K^{s-k}\cdot F^{(s-t)}\right\rangle=$$
$$=\delta_{r,s+1}\cdot q^{-2(1+h(s+1)+k(s+1))}\cdot(-1)^s$$

that is we have

$$\left\langle S\left(\overline{F}\otimes 1\right),E^{(r)}K^h\otimes K^kF^{(s)}\right\rangle=\delta_{r,s+1}\cdot q^{-2(1+h(s+1)+k(s+1))}\cdot(-1)^s\ ;$$

now, the element $-q^{-2}\cdot\sum_{n=0}^\infty\overline{F}^{n+1}L^{2(n+1)}\otimes L^{-2(n+1)}\overline{E}^n$ $\left(\in F_q^P[D]\right)$ when evaluated on PBW elements $E^{(r)}K^h\otimes K^kF^{(s)}$ of $D_q^Q(\mathfrak{g})$ takes the same values as $S\left(\overline{F}\otimes 1\right)$, therefore

$$S(\overline{F}\otimes 1)=-q^{-2}\cdot\sum_{n=0}^{\infty}\overline{F}^{n+1}L^{2(n+1)}\otimes L^{-2(n+1)}\overline{E}^n\ .$$

[9] As above, one can compute $\Delta\left(K^{\pm 1}\otimes 1\right)$, $\Delta\left(1\otimes L^{\pm 1}\right)$, and get $\Delta\left(F_q^Q[D]\right)\leq F_q^Q[D]\,\widehat{\otimes}\,F_q^Q[D]$.

With exactly the same technique we get formulas

$$S\left(L^{-1}\otimes L\right)=\sum_{n=0}^{\infty}\overline{F}^{n}L^{2n+1}\otimes L^{-2n-1}\overline{E}^{n}$$
$$S\left(L\otimes L^{-1}\right)=L^{-1}\otimes L-\overline{F}L\otimes L^{-1}\overline{E}$$
$$S\left(1\otimes\overline{E}\right)=-q^{+2}\cdot\sum_{n=0}^{\infty}\overline{F}^{n}L^{2(n+1)}\otimes L^{-2(n+1)}\overline{E}^{n+1}\ ;$$

by the way, this explicitly shows that $S\left(A_\infty^{0,P}\right)\leq A_\infty^{0,P}$ and[10] $S\left(\widetilde{A}_\infty^{0,P}\right)\leq\widetilde{A}_\infty^{0,P}$. In particular we directly proved that $A_\infty^{0,P}$ is a Hopf subalgebra of $F_q^P[D]$, and[11] $\widetilde{A}_\infty^{0,P}$ is a $k[q,q^{-1}]$-integer form of $A_\infty^{0,P}$.

The same procedure for $A_\infty^{0,Q}$ — formula (A.2) — gives

$$S\left(F^{(1)}\otimes 1\right)=-q^{-2}\cdot\sum_{n=0}^{\infty}\left(q-q^{-1}\right)^{2n}\cdot[n+1]_q!\cdot[n]_q!\cdot F^{(n+1)}K^{n+1}\otimes K^{-(n+1)}E^{(n)}$$
$$S\left(K^{-1}\otimes K\right)=\sum_{n=0}^{\infty}\left(q-q^{-1}\right)^{2n}\cdot[n]_q!^2\cdot[n+1]_q\cdot F^{(n)}K^{2n+1}\otimes K^{-2n-1}E^{(n)}$$
$$S\left(K\otimes K^{-1}\right)=K^{-1}\otimes K-[2]_q\cdot\left(q-q^{-1}\right)^2\cdot F^{(1)}\otimes E^{(1)}+\left(q-q^{-1}\right)^2\cdot F^{(1)}K\otimes K^{-1}E^{(1)}$$
$$S\left(1\otimes E^{(1)}\right)=-q^{+2}\cdot\sum_{n=0}^{\infty}\left(q-q^{-1}\right)^{2n}\cdot[n]_q!\cdot[n+1]_q!\cdot F^{(n)}K^{n+1}\otimes K^{-(n+1)}E^{(n+1)}\ ;$$

here again we remark that the involved series are convergent in the $(q-1)$-adic topology; moreover, we explicitly showed that $S\left(A_\infty^{0,Q}\right)\leq A_\infty^{0,Q}$ and[12] $S\left(\widehat{A}_\infty^{0,Q}\right)\leq\widehat{A}_\infty^{0,Q}$, thus $A_\infty^{0,Q}$ is a Hopf subalgebra of $F_q^Q[D]$ and[13] $\widehat{A}_\infty^{0,Q}$ is a $k[q,q^{-1}]$-integer form of $A_\infty^{0,Q}$.

Finally we record that — thanks to Theorem 6.2 — the substitutions

$$F\otimes 1\mapsto F\,,\qquad L^{\mp1}\otimes L^{\pm1}\mapsto L^{\pm1}\,,\qquad K^{\mp1}\otimes K^{\pm1}\mapsto K^{\pm1}\,,\qquad 1\otimes E\mapsto E$$

turn the above formulas defining the "Hopf operations" Δ, ε, and S of $A_\infty^{0,M}$ into analogous formulas defining Δ, ε, and S for $U_q^{M,\infty}(\mathfrak{h})$, for $M=P,Q$.

A.4. Let $\mathfrak{g}:=\mathfrak{sl}(2)$, and consider $U_q^Q(\mathfrak{g})=U_q^Q(\mathfrak{sl}(2))$; consider the 2-dimensional vector space V over $k(q)$ with (ordered) basis $\{v_+,v_-\}$: the standard representation $U_q^Q(\mathfrak{g})\to End_{k(q)}(V)$ is defined by

$$E\mapsto\begin{pmatrix}0&1\\0&0\end{pmatrix}\qquad K^{\pm1}\mapsto\begin{pmatrix}q^{\pm1}&0\\0&q^{\mp1}\end{pmatrix}\qquad F\mapsto\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

[10] Similarly one can compute $S\left(L^{\pm1}\otimes 1\right)$, $S\left(1\otimes L^{\pm1}\right)$, and so verify that $S\left(F_q^P[D]\right)\leq F_q^P[D]$.

[11] Similarly we could directly prove that $F_q^P[D]$ is a Hopf subalgebra of $D_q^Q(\mathfrak{g})^*$.

[12] Similarly we can also compute $S\left(K^{\pm1}\otimes 1\right)$, $S\left(1\otimes L^{\pm1}\right)$, and get $S\left(F_q^Q[D]\right)\leq F_q^Q[D]$.

[13] Similarly we could directly prove that $F_q^Q[D]$ is a Hopf subalgebra of $D_q^P(\mathfrak{g})^*$.

where matrices are relative to the ordered basis $\{v_+, v_-\}$; letting $\{\phi_+, \phi_-\}$ be the basis of V^* dual of $\{v_+, v_-\}$, we denote the matrix coefficients of this representation by

$$\begin{aligned} a &:= \langle \phi_+, -.v_+ \rangle \quad (x \mapsto \langle \phi_+, x.v_+ \rangle) \, , \qquad & b &:= \langle \phi_+, -.v_- \rangle \quad (x \mapsto \langle \phi_+, x.v_- \rangle) \\ c &:= \langle \phi_-, -.v_+ \rangle \quad (x \mapsto \langle \phi_-, x.v_+ \rangle) \, , \qquad & d &:= \langle \phi_-, -.v_- \rangle \quad (x \mapsto \langle \phi_-, x.v_- \rangle) \, . \end{aligned}$$

It is well known (cf. for instance [APW], Appendix, or [DL], §1) that the quantum function algebra $F_q^P[G] = F_q[SL(2,k)]$ is generated by the matrix coefficients of the standard representation, (this is true more in general for $G = SL(n+1,k)$) i. e. a, b, c, d. Thus $F_q^P[G]$ is the associative $k(q)$-algebra with 1 with generators a, b, c, d and relations

$$\begin{gathered} ab = qba \, , \qquad cd = qdc \, , \qquad ac = qca \, , \qquad bd = qdb \\ bc = cb \, , \qquad ad - da = (q - q^{-1}) \, bc \, , \qquad ad - qbc = 1 \, ; \end{gathered}$$

with Hopf algebra structure defined by formulas

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c \, , \qquad & \varepsilon(a) &= 1 \, , \qquad & S(a) &= d \\ \Delta(b) &= a \otimes b + b \otimes d \, , \qquad & \varepsilon(b) &= 0 \, , \qquad & S(b) &= -qb \\ \Delta(c) &= c \otimes a + d \otimes c \, , \qquad & \varepsilon(c) &= 0 \, , \qquad & S(c) &= -q^{-1}c \\ \Delta(d) &= c \otimes b + d \otimes d \, , \qquad & \varepsilon(d) &= 1 \, , \qquad & S(d) &= a \end{aligned} \tag{A.3}$$

and $\widehat{F}_q^P[G]$ is nothing but the $k[q,q^{-1}]$-subalgebra generated by a, b, c, d.

Similarly $F_q^P[B_+]$, resp. $F_q^P[B_-]$, has the following presentation: it is the associative $k(q)$-algebra with 1 with generators a_+, b_+, d_+, resp. a_-, c_-, d_-, and relations

$$\begin{aligned} & & a_+ b_+ &= q \, b_+ a_+ \, , \qquad & b_+ d_+ &= q \, d_+ b_+ \, , \qquad & a_+ d_+ &= 1 = d_+ a_+ \, , \\ &\text{resp.} & a_- c_- &= q \, c_- a_- \, , \qquad & c_- d_- &= q \, d_- c_- \, , \qquad & a_- d_- &= 1 = d_- a_- \, , \end{aligned}$$

with the Hopf algebra structure given by

$$\begin{aligned} \Delta(a_+) &= a_+ \otimes a_+ \, , \qquad & \varepsilon(a_+) &= 1 \, , \qquad & S(a_+) &= d_+ \\ \Delta(b_+) &= a_+ \otimes b_+ + b_+ \otimes d_+ \, , \qquad & \varepsilon(b_+) &= 0 \, , \qquad & S(b_+) &= -qb_+ \\ \Delta(d_+) &= d_+ \otimes d_+ \, , \qquad & \varepsilon(d_+) &= 1 \, , \qquad & S(d_+) &= a_+ \end{aligned}$$

resp.

$$\begin{aligned} \Delta(a_-) &= a_- \otimes a_- \, , \qquad & \varepsilon(a_-) &= 1 \, , \qquad & S(a_-) &= d_- \\ \Delta(c_-) &= c_- \otimes a_- + d_- \otimes c_- \, , \qquad & \varepsilon(c_-) &= 0 \, , \qquad & S(c_-) &= -q^{-1}c_- \\ \Delta(d_-) &= d_- \otimes d_- \, , \qquad & \varepsilon(d_-) &= 1 \, , \qquad & S(d_-) &= a_- \, ; \end{aligned}$$

and the integer form $\widehat{F}_q^P[B_+]$, resp. $\widehat{F}_q^P[B_-]$, is nothing but the $k[q,q^{-1}]$-subalgebra generated by a_+, b_+, d_+, resp. a_-, c_-, d_-.

The Hopf algebra epimorphisms

$$\rho_+ \colon F_q^P[G] \twoheadrightarrow F_q^P[B_+] \, , \qquad \rho_- \colon F_q^P[G] \twoheadrightarrow F_q^P[B_-]$$

given by restriction (cf. the proof of Theorem 5.7) are then described by

$$\rho_+ \colon a \mapsto a_+ \, , \; b \mapsto b_+ \, , \; c \mapsto 0 \, , \; d \mapsto d_+ \, , \qquad \rho_- : a \mapsto a_- \, , \; b \mapsto 0 \, , \; c \mapsto c_- \, , \; d \mapsto d_- \; .$$

A.5. We shall now explicitly realize the embedding of (formal) Hopf algebras (cf. §5)

$$\mu^P \colon F_q^P[G] \hookrightarrow A_\infty^P \, .$$

From the previous subsection it is easy to check that the Hopf algebra isomorphisms[14]

$$\vartheta_+ \colon F_q^P[B_+] \cong {U_q^P(\mathfrak{b}_-)}_{op} \;\; \text{(induced by } \pi_-) \, , \; \vartheta_- \colon F_q^P[B_-] \cong {U_q^P(\mathfrak{b}_+)}_{op} \;\; \text{(induced by } \overline{\pi_-})$$

(cf. (4.1)) are described by formulas

$$\vartheta_+ \colon a_+ \mapsto L^{-1} \, , \; b_+ \mapsto -\overline{F} L \, , \; d_+ \mapsto L \, , \qquad \vartheta_- : a_- \mapsto L \, , \; c_- \mapsto L^{-1} \overline{E} \, , \; d_- \mapsto L^{-1} \; .$$

As we saw in the proof of Theorem 5.7, μ^P is nothing but the composition

$$F_q^P[G] \xrightarrow{\Delta} F_q^P[G] \otimes F_q^P[G] \xrightarrow{\rho_+ \otimes \rho_-} F_q^P[B_+] \otimes F_q^P[B_-] \xrightarrow{\vartheta_+ \widehat{\otimes} \vartheta_-} {U_q^P(\mathfrak{b}_-)}_{op} \, \widehat{\otimes} \, {U_q^P(\mathfrak{b}_+)}_{op}$$

(whose image lies in A_∞^P) hence it is described by

$$\mu^P \colon \quad a \mapsto L^{-1} \otimes L - \overline{F} L \otimes L^{-1} \overline{E} \, , \;\; b \mapsto -\overline{F} L \otimes L^{-1} \, , \;\; c \mapsto L \otimes L^{-1} \overline{E} \, , \;\; d \mapsto L \otimes L^{-1} \, ;$$

then by direct comparison between formulas (A.3) and the formulas we gave for $F_q^P[D] :=$ $:= {U_q^P(\mathfrak{b}_-)}_{op} \otimes {U_q^P(\mathfrak{b}_+)}_{op}$ one immediately checks that μ^P preserves the (formal) Hopf structure, i. e. it is a monomorphism *of (formal) Hopf algebras*.

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[14] *Remark:* beware of the difference among these isomorphisms and the ones in [DL], §1, which are only *algebra* isomorphisms; for this reason the embedding $F_q^P[G] \hookrightarrow {U_q^P(\mathfrak{b}_-)}_{op} \otimes {U_q^P(\mathfrak{b}_+)}_{op}$ (see below) defined in [DL] is only an *algebra* monomorphism, while ours is a formal Hopf algebra monomorphism.

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Department of Mathematics
University of Rome "La Sapienza"
Piazzale Aldo Moro 5
00185 Rome — ITALY
e-mail: gavarini@mat.uniroma1.it